

DIRECT SUMS OF AELIAN GROUPS

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ABSTRACT:

In mathematics, the Seifert-van Kampen theorem of Algebraic topology, sometimes it is called as van Kampen's theorem. It expresses the structure of the fundamental group of a topological space X in terms of the fundamental groups of two open, path connected subspaces u and v that covers X is Abelian group.

KEYWORDS:

Abelian group , the external direct sum, direct sum, free abelian group, Rank of group.

INTRODUCTION:

Herbert karl Johannes Seifert & van Kampen introduced the problem of describing the fundamental group of a space X in terms of the fundamental groups of the constituents x_i of an open covering. In mathematics, the Seifert-van Kampen theorem of Algebraic topology, sometimes it is called as van Kampen's theorem. It expresses the structure of the fundamental group.

DEFINITION:

Let G be an abelian group and let $\{a_\alpha\}$ be an indexed family of elements of G ; let G_α be the subgroup of G generated by a_α .

If the groups G_α generate G , we also say that the elements a_α generate G . If each group G_α is infinite cyclic, and if G is the direct sum of the groups G_α ,

then G is said to be a **free abelian group** having the elements $\{a_\alpha\}$ as a **basis**

DEFINITION:

Let A be an additive abelian group and B, C two subsets of A . We write $B+C$ for the set $\{b+c; b \in B, c \in C\}$. $B+C$ is called the **sum** of B and C .

If $\{B_\alpha; \alpha \in I\}$ is an arbitrary collection of subsets of A , then the sum of $\{B_\alpha; \alpha \in I\}$ is defined to be the set of all finite sums $b_{\alpha_1} + \dots + b_{\alpha_n}$; $b_{\alpha_i} \in B, i=1,2,\dots,n, n \geq 1$.

DEFINITION:

suppose that the groups G_α generate G , and that for each $x \in G$, the expression $x = \sum x_\alpha$ for x is unique. That is, suppose that for each $x \in G$, there is only one J -tuple $(x_\alpha)_{\alpha \in J}$ with $x_\alpha = 0$ for all but finitely many α such that $x = \sum x_\alpha$. Then G is said to be the **direct sum** of the groups G_α , and we write

$$G = \bigoplus_{\alpha} G_\alpha,$$

Or in the finite case, $G = G_1 \oplus G_2 \oplus \cdots \oplus G_n$.

Example:

The cartesian product \mathbb{R}^ω is an abelian group under the operation of coordinate-wise addition. The set G_n consisting of those tuples (x_i) such that $x_i=0$ for $i \neq n$ is a subgroup isomorphic to \mathbb{R} . The groups G_n generate the subgroup \mathbb{R}^∞ of \mathbb{R}^ω ; indeed, \mathbb{R}^∞ is their direct sum.

DEFINITION:

Let $\{G_\alpha\}_{\alpha \in J}$ be an indexed family of abelian groups. Suppose that G is an abelian group, and that $i_\alpha: G_\alpha \rightarrow G$ is a family of monomorphisms, such that G is the direct sum of the groups (G_α) . Then we say that G is the **external direct sum** of the groups G_α , relative to the monomorphisms i_α .

DEFINITION:

If G is a free abelian group with a finite basis, the number of elements in a basis for G is called the **rank** of G .

THEOREM:

Let G be an abelian group; let $\{G_\alpha\}$ be a family of subgroups of G . If G is the direct sum of the groups G_α , then G satisfies the following condition:

Given any abelian group H and any family of

(*) homomorphisms $h_\alpha: G_\alpha \rightarrow H$, there exists a

homomorphism $h: G \rightarrow H$ whose restriction to G_α

equals h_α , for each α . Furthermore, h is unique. Conversely, if the groups G_α generate G and the extension condition (*) holds, then G is the direct sum of the groups G_α .

Proof:

We show first that if G has the stated extension property, then G is the direct sum of the G_α . Suppose

$x = \sum x_\alpha = \sum y_\alpha$; we show that for any particular index β , we have $x_\beta = y_\beta$.

Let H denote the group G_β ; and let $h_\alpha: G_\alpha \rightarrow H$ be the trivial homomorphism for $\alpha \neq \beta$, and the identity homomorphism for $\alpha = \beta$. Let $h: G \rightarrow H$ be the hypothesized extension of the homomorphisms h_α . Then

$$h(x) = \sum h_\alpha(x_\alpha) = x_\beta$$

$$h(x) = \sum h_\alpha(y_\alpha) = y_\beta,$$

so that $x_\beta = y_\beta$.

Now we show that if G is the direct sum of the G_α , then the extension condition holds. Given homomorphisms h_α , we define $h(x)$ as follows: If $x = \sum x_\alpha$, set $h(x) = \sum h_\alpha(x_\alpha)$.

Because this sum is finite, it makes sense; because the expression for x is unique, h is well-defined. One checks readily that h is the desired homomorphism. Uniqueness follows by noting that h must satisfy this equation if it is a homomorphism that equals h_α on G_α for each α .

THEOREM:

Given a family of abelian groups $\{G_\alpha\}_{\alpha \in J}$, there exists an abelian group G and a family of monomorphisms

$$i_\alpha: G_\alpha \rightarrow G \text{ Such that } G \text{ is the direct sum of the groups } i_\alpha(G_\alpha).$$

Proof:

Consider first the cartesian product

$$\prod_{\alpha \in J} G_\alpha;$$

$$\prod_{\alpha \in J} G_\alpha$$

it is an abelian group if we add two J -tuples by adding them coordinate-wise.

Let G denote the subgroup of the cartesian product consisting of those tuples $(x_\alpha)_{\alpha \in J}$ such that $x_\alpha = 0_\alpha$, the identity element of G_α , for all but finitely many values of α . Given an index β , define $i_\beta: G_\beta \rightarrow G$ by letting $i_\beta(x)$

be the tuple that has x as its β th coordinate and 0_α as its α th coordinate for all $\alpha \neq \beta$. It is immediate that i_β is a monomorphism.

It is also immediate that since each element x of G has only finitely many nonzero coordinates, x can be written uniquely as a finite sum of elements from the group (G_β) .

THEOREM:

Let $\{G_\alpha\}_{\alpha \in J}$ be an indexed family of abelian groups and G be an abelian group; let $i_\alpha: G_\alpha \rightarrow G$ be a family of homomorphisms. If each i_α is a monomorphism and G is the direct sum of the groups (G_α) , then G satisfies the following extension condition:

Given any abelian group H and any family of homomorphisms $h_\alpha: G_\alpha \rightarrow H$, there exists a homomorphism $h: G \rightarrow H$ such that $h \circ i_\alpha = h_\alpha$ for each α .

Furthermore, h is unique. Conversely, suppose the groups (G_α) generate G and the extension condition (*) holds. Then each i_α is a monomorphism, and G is the direct sum of the groups (G_α) .

Proof:

The only part that requires proof is the statement that if the extension condition holds, then each i_α is a monomorphism. That is proved as follows.

Given an index β , set $H = G_\beta$ and let $h_\alpha: G_\alpha \rightarrow G_\beta$ be the identity homomorphism if $\alpha = \beta$, and the trivial homomorphism if $\alpha \neq \beta$.

Let $h: G \rightarrow G_\beta$ be the hypothesized extension. Then in particular, $h \circ i_\alpha = h_\alpha$; it follows that i_α is injective

CONCLUSION:

In this dissertation we have discussed some basic definition. Also we have discussed the direct sum of abelian groups, Free Products of Groups and Free Groups. We also deals with the major theorem “The Seifert-van Kampen theorem” of the dissertation.

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