CONVERGENCE OF NET IN TOPOLOGICAL SPACE

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INTRODUCTION

The word topology is derived from Greek word "Topos" meaning "surface" and "Logos" meaning "Discourse" or "Study". Topology thus literally means the Study of Surfaces. Its English form "Topology" was introduced in 1883 within the journal nature to differentiate .

The term "net" was coined by John L. Kelly. Nets are one among the various tools utilized in topology to generalize certain concepts which will only be general enough within the context of metric spaces.

In mathematics, more specifically in general topology and related branches, a net or Moore-Smith sequence is a generalization of the notion of sequence. In essence, a sequence is a function with domain, the natural numbers and within the context of topology, the co-domain of this function is usually any topological space.

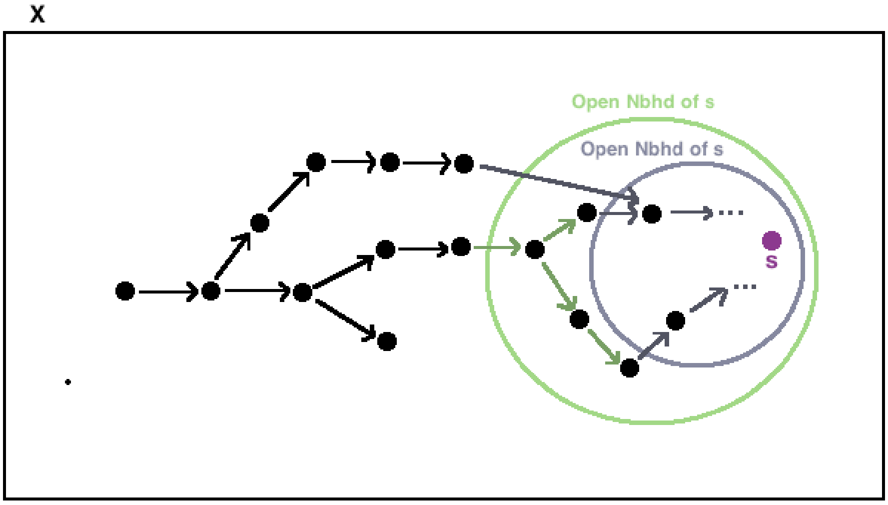
However, in the context of topology, sequences don't fully encode all information a few function between topological spaces.

The purpose of the concept of a net, first introduced by E. H. Moore and Herman L. Smith in 1922, is to generalize the notion of a sequence so as to confirm the equivalence of the conditions.

The purpose of this project is to provide through discussion of convergence of net in topological space and their properties.

DEFINITION:

A set X along side onτa topology it's called a mathematical space and is return as $(X, \square)$.



For any open neighbourhood of the point s, we can find a point N in the domain of the net such that when N is succeeded by n, the values S_n will be in that open neighbourhood of s.

DEFINITION:

Let X be a non – empty set and let $(D, \geq)$ be a directed set. Then, a mapping s: D ⟶ X is called a net in X.

DEFINITION:

A net during a mathematical space $)\tau(X,$ is claimed to converge to a point x ∈ X iff it is eventually in every neighborhood of x.

DEFINITION

Let $s = (\{sn : n \in D\}, \geq)$ and

$s^*=$ $(\{s_n{}^* : n \in E\}, \geq^*)$ be two nets in X. Then, thenet

$s^*$is called a subnet of s iff ∃ a mapping □of E into D such that

i. $s^*= s\ o\ \square and$

ii. for each $m \in D$ ∃ $n \in E$ suchthat

$p \geq^* n \Rightarrow m\ \forall\ p \in E.$

CONVERGENCE OF NET IN TOPOLOGICAL SPACE

DEFINITION

Let $(X,$ □$)$ be a mathematical space and $(f, X, D, \geq)$ be a net in X. The net f is said to be converge to a point $x \in X$ iff for each □$_1$-neighborhood N of x $\exists$ an element $a_0 \in D$ such that for every $a \in D$, $a \geq a_0$, $f_a \in N$. f is eventually in every □-neighborhood of x.

THEOREM

Let $(X,$ □$)$ be a mathematical space and Y □X then Y is

□-open iff no net in X-Y can converge to a point in Y.

PROOF

NECESSARY PART

Assume that no net in X-Y converges to a point in Y. To prove: Y is □-open. Suppose that Y is not a □-open. Then there exists a point $x_o$in Y such that every neighborhood of $x_o$intersects X-Y.

Let $N\ (x_0)$ be the collection of all □-neighborhoods of

x_0. Then $N(x_0)$ is directed by the inclusion relation □.

$Since\ N \cap (X - Y) \neq \emptyset\ \forall\ N \in N\ (x_0)$, for each

N $\in N(x_0)$, we may choose a $point\ x(N)\ in\ N \cap (X - Y)$.

Now consider the mapping

$$f: N(x_0) \rightarrow\ X - Y.$$

such that $f(N) = x(N) \forall N \in N(x)$. Then f is a netin

X-Y converging to x_0. To see that f converges to x_0, let V be any □-neighborhood of x_0..Then for each $W \in N(x_0)$ such that $W \geq V$, we have

$$f\ (W) = x(W) \in W \cap (X - Y) \text{□} V.$$

Thus there exists a net in X-Y converging to x_0.

This is contradiction. Hence Y must be open.

CONVERSE PART

Assume that Y is open.

Now, we have to prove that no net in X-Y can convergeto a point inY.

Then there exists a net $(f, X - Y, A, \geq)$ in X-Y converging to a point $x_0 \in$ Y. Since Y is a □-neighborhood of x_0, by the definition of convergence f must be eventually in Y. This means that f can never be in X-Y. But this is a contradiction.

Hence no net in X-Y can converge to a point in Y.

COROLLARY

A subset A of a topological space is closed iff no net in A converges to a point in X-A.

PROOF

Suppose that A is closed.

To prove

X-A is open.

i.e.) A is closed if and only if X-A is open.

Now choose Y = X-A.

And hence the above problem becomes the same as the previous theorem i.e. Y is open iff no net in X-Y converges to a point inY.

$$\therefore X - A \text{ is open.}$$

Hence A is closed.

THEOREM

A topological space $(X, \square)$ is hausdorff iff each net in X converges to at the most one point.

PROOF

Let $(X, \square)$ be hausdorff so that $x \neq y$ implies there exists neighborhoods N and M of x and y respectively such that

$$N \cap M \neq \square.$$

Since a net cannot be eventually in each of two disjoint sets therefore we conclude that no net in X can converge to both x and y.

Hence a net in X converge at the most to one point only.

CONVERSE PART

Let each net in X converge at the most to one point in X and we have to prove that $(X, \square)$ is Hausdorff.

Let if possible $(X, \square)$ be not Hausdorff so that there exist distinct points x and y such that each neighborhood of y. If $N_{\infty}$and N_{v}denote the collection of all $\square$-neighborhoods of x and y respectively then we know that

$$(N_{\square}, \square), (N_{v}, \in) \ \textit{are directed sets}.$$

Now consider the product set P= $N_{\square} \times N_{v}$ and let us define a relation $\geq$ in P as follows.

Let $(N_1, M_1), (N_2, M_2)$ be any two element of P. Then$(N_1, M_1) \geq (N_2, M_2)$ iff $N_1 \square N_2$ $and M_1 \square M_2$.

We can easily verify that P is directed by relation $\geq$

defined as above.

As every neighborhood of x intersects neighborhood of y,therefore

$$N \cap M = \square \forall (T, M) \in P.$$

Thus we may choose a point

$x_{(N,M)}$ *in* N ∩ □ ∀ (□, □) ∈ □.

Let us now consider the mapping

□: □ → □ : □(□, □) ∈ □.

□□□ □□□□□ (□, □) ∈ □.

We shall prove below that f is a net in X which converges to both *x* and y.

Let U,V be any neighborhoods of *x* and y respectively then for each (N, M) in P such that

(□, □) ≥ (□, □)[□. □. □□□, □□□].

We have □ (□, □) = □(□,□) ∈ □ ∩ □□□ ∩ □.

Above relation shows that f(N, M)∈ U and f(N, M)∈ V.

Thus we conclude that f is eventually in both U and V and so f converges to both *x* and y.

This is a contradiction.

Hence X must be a hausdorff space.

THEOREM

Let (X,□) be a hausdorff topological space. Then every convergent net has a unique cluster point and this is the unique limit point of the net.

PROOF

We know that in hausdorff space, every convergent net has a unique limit point. Let p be the unique limit point of a convergent net f in X. Since every limit point is also a cluster point, p is a cluster point of f. Suppose, if possible, f has another cluster point q distinct from p. Since X is a T_2-space there exists disjoint neighborhood □$_□$of p and q respectively. Since p is the limit of the net f, f is eventually in □$_□$ and since □$_□$ is disjoint from □$_□$, f cannot be frequently in □$_□$.

This contradicts our assumption that q is a cluster point of f.

Hence f cannot have two distinct cluster points.

∴ Every convergent net in X has a unique cluster point.

THEOREM

A topological space (□, □) is hausdorff space iff every net in x can converge to at most one point.

PROOF

Let (X,□) be a hausdorff space and let x, y be any two distinct points of X .Then there exists a neighbourhoods M and N of x and y such that □ ∩ □ = □. Since a net can never be eventually in each of the two disjoint sets, it follows that no net in X can converge to both x and y.

Hence a net in X can converge to at most one point ofX.

Conversely, let (X,□) be a topological space and let every net in X converge to at most one point of X. Then we must show that X is hausdorff. If possible, let it be nothausdorff.

Then there exists two distinct points x and y of X such that x and y do not have disjoint neighborhoods. In other words, every neighborhood of x intersects every neighborhood of y.

Let $\mathbf{N}_x$and $\mathbf{N}_y$be the families of all neighborhoods of x and y respectively. Then ($N_{x,}$□) as well as ($\mathbf{N}_{y,}$□) is clearly a directed set.

Now, on $\mathbf{N}_x \times \mathbf{N}_y$, we define a relation ≥ as under:

If (M_1, N_1) and (M_2, N_2) be any two members of $\mathbf{N}_x \times \mathbf{N}_y$

then

$(M_1, N_1) \geq (M_2, N_2)$ □M_1□M_2&N_1□N_2.

The system ($\mathbf{N}_x \times \mathbf{N}_y$, ≥) is then clearly a directed system. Since each neighborhood of x intersects each neighborhood of y, it follows that corresponding to each pair (M, N) of neighborhoods M & N of x and y respectively, we may choose a point $x_{(M,N)}$in M ∩ N.

Now, consider the mapping,

$s : \mathbf{N}_x \times \mathbf{N}_y \rightarrow X : s(M, N) = x_{(M,N)} \forall \ (M, N) \in \mathbf{N}_x \times \mathbf{N}_y$.

Then, s is clearly a net in X.

We claim that s converges to both x and y.

Let V and W be neighborhoods of x and y respectively,

then,

$$(M, N) \in \mathbf{N}_x \times \mathbf{N}_y, (M, N) \geq (V,W)$$

$$\Rightarrow s(M, N) = x_{(M,N)} \in M \cap N \ \square V \cap W.$$

$$[\text{Since } (M, N) \geq (V,W) \Rightarrow M \ \square V \text{ and } N \ \square W]$$

$$\Rightarrow s(M, N) \in V \text{ and } s(M, N) \in W.$$

This shows that s is eventually in V as well as in W.

But V and W being the arbitrary neighborhoods of x and y respectively, so it follows that s converges to both x andy.

This is a contradiction.

Hence, (X,□) is a Hausdorff space.

CONCLUSION

In this desertion we discussed about related topics about convergence of net basic definition and result of convergence used in theorems. Then the convergence of net play fundamental role in the topological space.

The concept of convergence of net is explained lucidly with the notion of sequence and physical applications.

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