# DIRECT SUMS OF ABELIAN GROUPS

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**ABSTRACT:**

In mathematics, the Seifert-van Kampen theorem of Algebraic topology, sometimes it is called as van Kampen's theorem. It expresses the structure of the fundamental group of a topological space X in terms of the fundamental groups of two open, path connected subspaces u and v that covers X is Abelian group.

**KEYWORDS:**

Abelian group , the external direct sum, direct sum, free abelian group, Rank of group.

**INTRODUCTION:**

Herbert karl Johannes Seifert & van Kampen introduced the problem of describing the fundamental group of a space X in terms of the fundamental groups of the constituents $x_i$ of an open covering.In mathematics, the Seifert-van Kampen theorem of Algebraic topology, sometimes it is called as van Kampen's theorem. It expresses the structure of the fundamental group.

**DEFINITION:**

Let G be an abelian group and let $\{a_\alpha\}$ be an indexed family of elements of G; let $G_\alpha$be the subgroup of G generated by $a_\alpha$.

If the groups $G_\alpha$generate G, we also say that the elements $a_\alpha$generate G. If each group $G_\alpha$is infinite cyclic, and if G is the direct sum of the groups $G_\alpha$, then G is said to be a **free abelian group** having the elements $\{a_\alpha\}$ as a **basis**

**DEFINITION:**

Let A be an additive abelian group and B,C two subsets of A. We write B+C for the set {b+$c$;b ∈ B,c ∈ C}. B+C is called the **sum** of B and C. If $\{B_\alpha : \alpha \in I\}$ is an arbitrary collection of subsets of A, then the sum of $\{B_\alpha : \alpha \in I\}$ is defined to be the set of all finite sums $b_{\alpha_1} + \cdots + b_{\alpha_n}$; $b_{\alpha_1} \in$ B, i=1,2,⋯,n, n ≥1.

**DEFINITION:**

suppose that the groups $G_\alpha$generate G, and that for each x∈G, the expression $x = \sum x_\alpha$for x is unique. That is, suppose that for each x $\in G$, there is only one J-tuple $(x_\alpha)_{\alpha}$□J with $x_\alpha$=0 for all but finitely many $\alpha$such that x=$\sum x_\alpha$. Then G is said to be the **direct sum** of the groups $G_\alpha$, and we write

$$G = \bigoplus_{\alpha \in J} G_\alpha,$$

Or in the finite case, $G = G_1 \oplus G_2 \oplus \cdots \oplus G_n$.

**Example:**

The cartesian product $\mathbb{R}^{\omega}$ is an abelian group under the operation of coordinate-wise addition. The set $G_n$ consisting of those tuples $(x_i)$ such that $x_i=0$ for $i \neq n$ is a subgroup isomorphic to $\mathbb{R}$. The groups $G_n$ generate the subgroup $\mathbb{R}^{\infty}$ of $\mathbb{R}^{\omega}$; indeed, $\mathbb{R}^{\infty}$ is their directsum.

**DEFINITION:**

Let $\{G_\alpha\}_{\alpha}$□J be an indexed family of abalian groups. Suppose that G is an abelian group, and that $i_\alpha: G_\alpha \to G$ is a family of monomorphisms, such that G is the direct sum of the groups $(G_\alpha)$. Then we say that G is the **external direct sum** of the groups $G_\alpha$, relative to the monomorphisms $i_\alpha$.

**DEFINITION:**

If G is a free abelian group with a finite basis, the number of elements in a basis for G is called the **rank** of G.

**THEOREM:**

Let G be an abelian group; let $\{G_\alpha\}$ be a family of subgroups of G. If G is the direct sum of the groups $G_\alpha$, then G satisfies the following condition:

Given anyabelian group H and any family of (∗)homomorphisms $h_\alpha: G_\alpha \to H$, there exists a homomorphism $h: G \to H$ whose restriction to $G_\alpha$ equals $h_\alpha$, for each $\alpha$. Furthermore, h is unique. Conversely, if the groups $G_\alpha$ generate G and the extension condition (∗) holds, then G is the direct sum of the groups $G_\alpha$.

**Proof:**

WeshowfirstthatifGhasthestatedextensionproperty, then G is the direct sum of the $G_\alpha$. Suppose

$x$=$\sum x_\alpha$=$\sum y_\alpha$;we show that for any particular index $\beta$, we have $x\beta$=$y\beta$.

Let H denote the group $G\beta$;and let $h_\alpha$:$G_\alpha \rightarrow H$be the trivial homomorphism for $\alpha \neq \beta$,and the identity homomorphism for $\alpha = \beta$.Let h : G → H be the hypothesized extension of the homeomorphisms' $h_\alpha$. Then

$$\text{h}(x)=\sum h_\alpha(x_\alpha)=x\beta$$

$$\text{h}(x)= \sum h_\alpha(y_\alpha)=y\beta,$$

so that $x\beta$=$y\beta$.

Now we show that if G is the direct sum of the $G_\alpha$, then the extension condition holds. Given homomorphisms $h_\alpha$, we define h($x$) as follows: If $x = \sum x_\alpha$, set h($x$)=$\sum h_\alpha(x_\alpha)$.

Because this sum is finite, it makes sense; because the expression for $x$is unique, h is well-defined. One checks readily that h is the desired homomorphism. Uniqueness follows by nothing that h must satisfy this equation if it is a homomorphism that equals $h_\alpha$on $G_\alpha$for each $\alpha$.

**THEOREM:**

Given a family of abelian groups $\{G_\alpha\}_{\alpha \in J}$, there exists an abelian group G and a family of monomorphisms

$i_\alpha : G_\alpha \rightarrow$ G Such that G is the direct sum of the groups $i_\alpha(G_\alpha)$.

**Proof:**

Consider first the cartesian product

$$\prod_{\alpha \in J} G_\alpha;$$

it is an abelian group if we add two J-tuples by adding them coordinate-wise.

Let G denote the subgroup of the cartesianproductconsisting of those tuples $(x_\alpha)_{\alpha \in J}$ such that $x_\alpha = 0_\alpha$,theidentity element of $G_\alpha$, for all but finitely many values of $\alpha$.Given an index $\beta$,define $i_\beta : G_\beta \rightarrow G$by letting $i_\beta(x)$
be the tuple that has $x$as its $\beta$th coordinate and $0_\alpha$as its $\alpha$th coordinate for all $\alpha \neq \beta$. It is immediate that $i_\beta$is a monomorphim.

It is also immediate that since each element x of G has only finitely many nonzero coordinates, x can be written uniquely as a finite sum of elements from the group $(G_\beta)$.

**THEOREM:**

Let $\{G_\alpha\}_{\alpha}$□J be an indexed family of abelian groupsletG be an abelian group; let $i_\alpha$: $G_\alpha \rightarrow G$be a family of homomorphisms. If each $i_\alpha$is a monomorphism and G is thedirect sum of the groups ($G_\alpha$), then G satisfies the following extensioncondition:

Givenany abelian group Hand any of(∗) homomorphisms $h_\alpha$:$G_\alpha \rightarrow H$, there exists a homomorphism h : G → $H$such that $h \circ i_\alpha = h_\alpha$for each $\alpha$.

Furthermore, h is unique. Conversely, suppose the groups ($G_\alpha$) generate G and the extension condition (∗) holds. Then each $i_\alpha$is a monomorphism, and G is the direct sum of the groups ($G_\alpha$).

**Proof:**

The only part that requires proof is the statement that if the extension condition holds, then each $i_\alpha$is a monomorphism. That is proved asfollows.

Given an index $\beta$, set H = □□and let $h$□: □□→ □be the identity homomorphism if □ = □,and the trivial homomorphism if □ ≠□.

Let h : G → □be the hypothesized extension. Then in particular, h ∘□□= $h$□; it follows that □□is injective

**CONCLUSION:**

In this dissertation we have discussed some basic definition. Also we have discussed the direct sum of abelian groups, Free Products of Groups and Free Groups. We also deals with the major theorem "The Seifert-van Kampen theorem" of the dissertation.

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