THE SEIFERT-VAN KAMPEN THEOREM

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ABSTRACT:

Seifert & van Kampen introduced the problem of describing the fundamental group of a space X in terms of the fundamental groups of the constituents x_i of an open coveringIn mathematics, the Seifert-van Kampen theorem of Algebraic topology, sometimes it is called as van Kampen's theorem. It expresses the structure of the fundamental group.

KEYWORDS:

Trivial topology ,Topological space, open set, discrete topology, subspace topology.

INTRODUCTION:

In mathematics, the Seifert-van Kampen theorem of Algebraic topology, sometimes it is called as van Kempen's theorem. It expresses the structure of the fundamental group of a topological space X in terms of the fundamental groups of two open, path connected subspaces u and v that covers X. One can use van Kampan's theorem to calculate fundamental groups for topological spaces that can be decomposed into simpler spaces.

DEFINITION:

A topology on a set X is a collection $\tau$of subsets of X having the following properties:

(1) $\emptyset$and X are inτ.

(2) The union of the elements of any subcollectionof

$\tau$is in τ.

(3) The intersection of the elements of any finite subcollection of $\tau$is inτ.

A set X for which a topology τhas been specified is called a topological space.

EXAMPLE:

Let X={a, b, c}.Then this set has 2^3 elements.Then,

τ={{},{a},{b},{c},{a,b},{b,c},{a,c},{a,b,c}}

To verify that $\tau$is a topology on X or not.

Axioms:1

∅and X are in τ.

Axioms:2

$\emptyset \cup \{a\} = \{a\} \in \tau$

$\emptyset \cup \{b\} = \{b\} \in \tau$

$\emptyset \cup \{c\} = \{c\} \in \tau$

$\{a\} \cup \{a, b\} = \{a, b\} \in$

$\emptyset \cup X = X \in \tau$

Axioms:3

$\emptyset \cap \{a\} = \emptyset \in \tau$

$\emptyset \cap \{b\} = \emptyset \in \tau$

$\emptyset \cap \{c\} = \emptyset \in \tau$

$\{a\} \cap \{a, b\} = \{a\} \in \tau$

All the three axioms are satisfied.

Therefore $\tau$is a topology.

DEFINITION:

If a set contains only one element it is calleda
singleton set.

Example:

A={0} , B={0} and the set of all even primes are all singleton sets.

DEFINITION:

A set B is called a subset of A if every element of B is
in A.

Example:

The set of all vowels is a subset of the set of all letters in English alphabet.

DEFINITION:

If X is any set, the collection of all subsets of X is a topology on X; it is called the discrete topology.The collection consisting of X and ∅ only is also a topology on X; we shall call it the indiscrete topology or the trivial topology.

THEOREM:

Given a family of abelian groups $\{G_\alpha\}_{\alpha \in J}$, there exists an abelian group G and a family of monomorphisms
$i_\alpha: G_\alpha \rightarrow G$ Such that G is the direct sum of the groups
$i_\alpha(G_\alpha)$.

Proof:

Consider first the cartesian product $\prod_{\alpha \in J} G_\alpha$;

it is an abelian group if we add two J-tuples by adding them coordinate-wise.

Let G denote the subgroup of the cartesian productconsisting of those tuples $(x_\alpha)_{\alpha \in J}$ such that $x_\alpha = 0_\alpha$,theidentity element of G_α, for all but finitely many values of α.Given an index β,define i_β: $G_\beta \rightarrow G$by letting $i_\beta(x)$ be the tuple that has $x$as its βth coordinate and 0_αas its αth coordinate for all $\alpha \neq \beta$. It is immediate that $i_\beta$is a monomorphism.It is also immediate that since each element x of G has only finitely many nonzero coordinates, x can be written uniquely as a finite sum of elements from the group (G_β).

LEMMA:

Let $\{G_\alpha\}_{\alpha}$□J be an indexed family of abelian groups; let G be an abelian group; let i_α: $G_\alpha \rightarrow G$be a family of homomorphisms. If each $i_\alpha$is a monomorphism and G is the direct sum of the groups (G_α), then G satisfies the following extensioncondition:

Given any abelian group H and any family of(∗) homomorphisms h_α:$G_\alpha \rightarrow H$, there exists a homomorphism h : G → Hsuch that $h \circ i_\alpha = h_\alpha$for each α.

Furthermore, h is unique. Conversely, suppose the groups (G_α) generate G and the extension condition (∗) holds. Then each $i_\alpha$is a monomorphism, and G is the direct sum of the groups (G_α).

Proof:

The only part that requires proof is the statement that if the extension condition holds, then each $i_\alpha$is a monomorphism. That is proved asfollows.

Given an index β, set H = $G_\beta$and let h_α: $G_\alpha \rightarrow H$be the identity homomorphism if $\alpha = \beta$,and the trivial homomorphism if $\alpha \neq \beta$.

Let h : G → Hbe the hypothesized extension. Then in particular, h $\circ i_\beta = h_\beta$; it follows that $i_\beta$is injective.

The Seifert-van Kampen Theorem

THEOREM:

Suppose X = U∪V, where U and V are open sets of X. Suppose U∩V is path connected, and that $x_0 \in U \cap V$. Let i and j be the inclusion mappings of U and V, respectively, into X. Then the images of the induced homomorphism

$$i_*: \pi_1(U,x_0) \rightarrow \pi_1(X,x_0) \text{ and } j_*: \pi_1(V,x_0) \rightarrow \pi_1(X,x_0)$$

generate $\pi_1(X,x_0)$.

Proof:

This theorem states that, given any loop f in X based at x_0, it is path homomorphic to a product of the form $(g_1*(g_2*(\ldots*g_n)))$,where each $g_i$is a loop in X based at x_0 that lies either in U or inV.

Step 1:

We show there is a subdivision $a_0<a_1<\cdots<a_n$ of the unit interval such that $f(a_i) \in U\cap V$ and $f([a_{i-1},a_i])$ is contained either in U or in V, for each i.

To begin, choose a subdivision $b_0,\ldots,b_m$ of [0,1] such that for each i, the set $f([b_{i-1},b_i])$ is contained in either U or

If $f(b_i)$ belongs to U∩V for each i, we are finished. If not, let i be an index such that $f(b_i) \notin U\cap V$. Each of thesets

$f([b_{i-1},b_i])$ and $f([b_i ,b_{i+1}])$ lies either in U or in V. If $f(b_i) \in U$, then both of these sets must lie in U; and if $f(b_i) \in V$, both or then must lie in V. In either case, we may delete b_i, obtaining a new subdivision $c_0,\ldots,c_{m-1}$ that still satisfies the condition that $f([c_{i-1},c_i])$ is contained either in U or in V, for each i.

A finite number of repetitions of this process leads to the desired subdivision.

Step 2:

We prove the theorem. Given f, let $a_0,\ldots,a_n$ be the subdivision in Step 1. Define f_i to be the path in X that equals the positive linear map of [0,1] onto $[a_{i-1},a_i]$ followed by f. Then f_i is a path that lies either in U or in V,and

$$[f] = [f_1]*[f_2]* \cdots *[f_n].$$

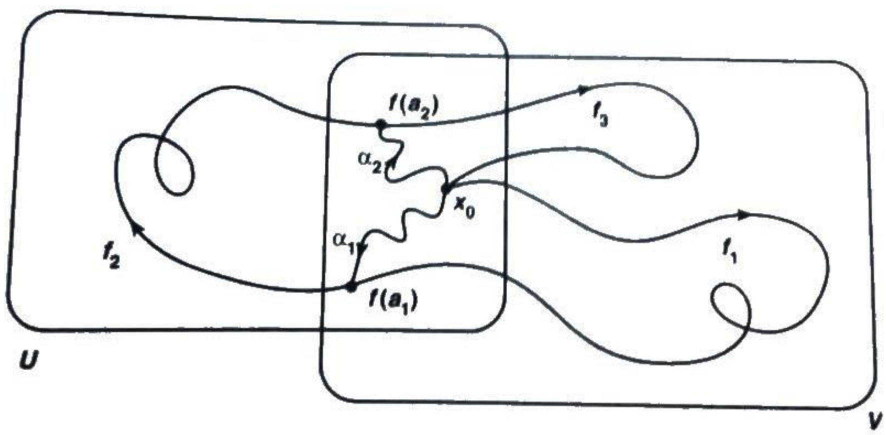
For each i, choose a path $\alpha_i$in U∩V from x_0 to $f(a_i)$. (Here we use fact that U∩V is path connected.) Since $f(a_0) = f(a_n) = x_0$, we can choose α_0 and $\alpha_n$to be the constant path at x_0.

Now we set

$$g_i=(\alpha_{i-1}*f_i)*\bar{\alpha}_i$$

for each i. Then $g_i$is a loop in X based at x_0 whose image lies either in U or in V. Direct computation showsthat

$$[g_1]*[g_2]* \cdots *[g_n] = [f_1]*[f_2]* \cdots *[f_n].$$



CONCLUSION:

A Seifert-van Kampen theorem is apparently applied to describe the Kumerian fundamental group of a semistable curve as the fundamental group of a graph ofgroups

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