A STUDY ON FOUR SQUARE THEOREM

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ABSTRACT:

The four square theorem was proved by Lagrange in 1770; every positive integer is the sum of four squares. i.e, n=$A^2+B^2+C^2+D^2$, where, A,B,C,D ϵ z An interesting proof is presented here based on Hurwitz integers, a subset of Quarternions which act like integers in four dimension and have the prime divisor property.

KEYWORDS:

Geometric Number, probabilities number, analytical number theory.

INTRODUCTION:

Number theory, known as the queen of mathematics is the branch of mathematics that concerns about the positive integers 1, 2, 3, 4, 5 which are often called natural numbers and their appealing properties from antiquity, these natural numbers classified as odd numbers, even numbers, square numbers, prime numbers, Fibonacci numbers, triangular numbers, etc. Due to the dense of unsolved problems, number theory plays a significant role in mathematics.

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DEFINITION:

QUARDRATIC FORM:

A binary quardratic form$Ax^2 + Bxy + cy^2$, Where A,B,C∈ $\mathbb{Z}$, can be viewed as an integer –valued function of integer pairs, or vectors (x, y).

Example, $2x^2 + 3xy + 4y^2$

Where, A= 2 ,B= 3, C=4

DEFINITION:

HURWITZ INTEGER:

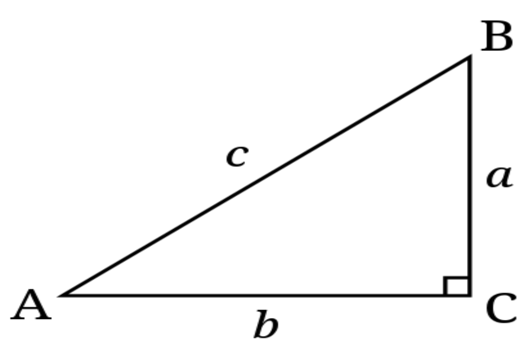
The quaternions in $\mathbb{Z}\,[\frac{1+i+j+k}{}$,i ,j ,k] are called the Hurwitz

Integers.

DEFINITION:

DIOPHANTINE EQUATION:

A equation of the form $a^2 + b^2 = c^2$ where a, b and c are integers is known as a diophantine equation.



FOUR SQUARE THEOREM:

DEFINITION:

Every natural number can be represented as the sum of four integer square.

$$P = a_0^2 + a_1^2 + a_2^2 + a_3^2$$

Where the four numbers a_0 , a_1 , a_2, a_3 are integers. For 1 , 2 ,3 , 4 can be represented as the sum of four square theorem.

EXAMPLES

$$1 = 1^2 + 0^2 + 0^2 + 0^2$$

$$2 = 1^2 + 1^2 + 0^2 + 0^2$$

$$3 = 1^2 + 1^2 + 1^2 + 0^2$$

$$4 = 1^2 + 1^2 + 1^2 + 1^2 \quad \text{(or)} \quad 2^2 + 0^2 + 0^2 + 0^2$$

REAL MATRICES AND C :

In this chapter we introduce 4 –dimensional "Hypercomplex numbers" called quaternion. A quaternion is easily defined as a 2×2 matrix of complex numbers , but to see why we might expect matrices to behave like numbers, we first show to model the complex numbers by 2×2 real matrices.

For each $a + bi \in \mathbb{C}$, with real a and b, consider the matrix

$$M(a + bi) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

It is easy to check that

$$M(a_1 + b_1 i) + M(a_2 + b_2 i) = M((a_1 + a_2 + (b_1 + b_2)i))$$

$$= M((a_1 + b_1 i) + (a_2 + b_2 i))$$

$$M(a_1 + b_1 i)\ M(a_2 + b_2 i) = M (a_1 a_2 - b_1 b_2 + (a_1 b_2 + b_1 a_2)i)$$

$$= M((a_1 + b_1 i)(a_2 + b_2 i)).$$

Thus matrix sum and product correspond to complex sum and product , and therefore the matrices

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \quad \text{for } a,b \in \mathbb{R}$$

Behave exactly like the complex numbers a + bi.

Another way to see this is to write

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = a1 + bi.$$

The identity matrix ,

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Behaves like the number 1 , and

$$i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Behaves like $\sqrt{-1}$. Indeed

$$i^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -1.$$

Not only does this matrix representation of $\mathbb{C}$ have natural counterparts of 1 and i , it also has a natural interpretation of the norm on $\mathbb{C}$ as the determinant . this is so because

$$\text{norm}(a + bi) = a^2 + b^2 = \det \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

The multiplicative property of the norm follows from the multiplicative property of the determinant;

$$\det\begin{pmatrix} a1 & b1 \\ -b1 & a1 \end{pmatrix} \det\begin{pmatrix} a2 & b2 \\ -b2 & a2 \end{pmatrix} =$$

$$\mathbf{det}\left(\begin{pmatrix} a1 & b1 \\ -b1 & a1 \end{pmatrix}\begin{pmatrix} a2 & b2 \\ -b2 & a2 \end{pmatrix}\right) \quad \rightarrow \ \mathbf{(*)}$$

And since the matrix product on the right-hand side equals

$$\begin{pmatrix} 1a2 - b1b2 & a1b2 + b1a2 \\ -a1b2 - b1a2 & a1a2 - b1b2 \end{pmatrix}$$

Equation (*) gives a new way to derive the diophantus two square identity .

Replacing each $\det\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ in (*) by $a^2 + b^2$ we get

$$(a_1^2 + b_1^2)(a_2^2 + b_2^2) = (a_1a_1 - b_1b_2)^2 + (a_1b_2 + b_1a_2)^2.$$

THE FOUR SQUARE IDENTITY:

If q = a1 + bi + cj + dk then norm(q) is

$$\det\begin{pmatrix} a + di & b + ci \\ -b + ci & a - di \end{pmatrix} = a^2 + b^2 + c^2 + d^2$$

since $\det(q_1)\det(q_2) = \det(q_1q_2)$ we can also write the “complex two square identity” as a real four square identity . which turns out to be

$$(a^2_1 + b^2_1 + c^2_1 + d^2_1)(a^2_2 + b^2_2 + c^2_2 + d^2_2) =$$

$$(a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2)^2$$

$$+(a_1b_2 - b_1a_2 + c_1d_2 - d_1c_2)^2$$

$$+(a_1c_2 - b_1d_2 + c_1a_2 + d_1b_2)^2$$

$$+(a_1d_2 + b_1c_2 - c_1b_2 + d_1a_2)2$$

Remarkably , the four square identity was discovered by Euler in 1748, nearly 100 years before the discovery of quaternion. Euler hoped to use it to prove that every natural number is the sum of four squares, by proving also that every prime is the sum of four squares.

This was first proved by Lagrange in 1770. We can now give a simpler proof with the help of quaternion.

$\mathbb{Z}[\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}]$:

From now on we write the quaternion 1 simply as 1 and omit it altogether as a item in a product . thus the typical quaternion will be written.

$$q = a + bi + cj + dk,$$

where $a , b , c , d \in \mathbb{R}$.

Which of these objects should be regarded as "integer"?

One 's first thought is that

$$\mathbb{Z}[i , j , k] = \{ a + bi + cj + dk: \ a , b , c , d \in \mathbb{Z} \}$$

Should be the "quaternion integer" analogous to the Gaussian integers $\mathbb{Z}[i]$. Sum , difference and product of members $\mathbb{Z}[i , j , k]$ are

again members of $\mathbb{Z}[i , j , k]$ and

$$\text{Norm}(a + bi + cj + dk) = a^2 + b^2 + c^2 + d^2$$

Is an ordinary integer, which we can use to find "primes" in $\mathbb{Z}[i , j , k]$.

EXAMPLE:

$2 + i + j + k$ is a prime of $\mathbb{Z}[i , j , k]$.

SOLUTION;

This is so because norm ,

$$(2 + i + j + k) = 2^2 + 1^2 + 1^2 + 1^2$$

$$=7$$

Which is a prime in $\mathbb{Z}$, hence 2 + i + j + k is not the product of members of $\mathbb{Z}[i , j , k]$with smaller norm.

HURWITZ INTEGER:

The quaternion in $\mathbb{Z}\left[\frac{1+i+j+k}{2}, i, j, k\right]$ are called the Hurwitz integers.

CONCLUSION:

In this paper discussed about number theory using Pell equation. Also we have discussed Brahmagupta composition rule , the quaternion units , the Hurwitz integers , conjugates , a prime divisor property , and four square theorem.

As well as the well – known applications to many others areas of mathematics.

Medical science is not an exact science in which processes can be easily analysed and modelled. These are all the field of current research in number theory.

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