A STUDY ON HOMOTOPY THEORY

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ABSTRACT:

wearepresentedwithtwospaceXandYtheproblemofdecidingwhetherornottheyarehomeomorphicisformidable.WehaveeithertoconstructahomeomorphismbetweenXandYor,worsestill,toprovethatnosuchhomeomorphismexists.Wetheretrytoreflecttheproblemalgebraically.

INTRODUCTION:

For the sake of clarity, if we are presented with two space X and Y the problem of deciding whether or not they are homeomorphic is formidable. We have either to construct a homeomorphism between X and Y or, worse still, to prove that no such homeomorphism exists. We there try to reflect the problem algebraically.

DEFINITION:

Tow continuous maps $f_{0,}f_1: X \to Y$ are said to de homotopic if ther is a continuous map F: $X \times C \to Y$ such that for every $x \in X$.

$F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$

The map F is called **homotopy** between f_0 and f_1.

EXAMPLE:

Let $X = y = R^n$ and let $f_0(x) = x \; and \; f_1(x) = 0 \; for \; x \in R^n$. Let $F: R^n \times C \to R^n$ be defined by

$$F(x, t) = (1 - t)x.$$

Then f is homotopy between f_0 and f_1.

DEFINITION:

A continuous map $f: X \longrightarrow Y$ is said to be **null homotopic** if it is homotopic to some constant map.

DEFINITION:

The equivalence class under $\cong$ of f is denoted by $[f]$,and is called the **homotopy class** of f.

DEFINITION:

A homotopy from 1_x to the constant map of X to $x_0 \in X$is called a **contraction** of X to x_0

DEFINITION:

The mappings f and g are called **homotopy equivalence.** The spaces X and Y are also called **homotopy.**

THEOREM:

If Y is contractible, then every continuous mapping $f: X \rightarrow Y$ is homotopy to a constant.

Proof:

Since y is contractible, there exists a continuous mapping

$F: Y \times C \rightarrow Y$ with $F(y, 0) = y$ and $F(y, 1) = y_0$ fro every $y \in Y$,where y_0 is a fixed element.

Let $f: X \rightarrow Y$ be a continuous mapping.

Define $G: X \rightarrow Y$ by

$G(x, t) = F(f(x), t)$ for $x \in X$.

G is continuous.

$$G(x,0) = F(f(x),0) = f(x)$$

$G(x,1) = F(f(x),1) = y_{0.}$

This show that f is homotopy to $g: X \to Y$ where g is a constant mapping defined by $g(x) = y_0$ for every $x \in X$.

THEOREM:

The relation of being of the same homotopy type is an equivalence relation.

Proof:

Reflexivity and symmetry follow easily from the definition.

Let X, Y be the same homotopy type as Y and Z respectively.

There exist continuous mappings $f: X \to Y$, $G: Y \to X, f_1: Y \to Z, g_1: Z \to Y$ such that $gf, fg, g_1f_1$and f_1g_1 are homotopy to the appropriate identity mappings.

Consider the transformations

$f_2: X \to Z$ and $g_2: Z \to X$

Defined by

$f_2(x) = f_1(f(x))$ and $g_2(x) = g\big(g_1(x)\big)$.

If $f_2$andg_2 are continuous and $g(g_1f_1)$ is homotopy to g, because (g_1f_1) is homotopic to the identity.

So, $g_2f_2 = g((g_1f_1)f)$ is homotopy to $gf$and which ultimately gives that g_2f_2 is homotopy to the identity.

Similarly, f_2g_2 is homotopy to the identity $I: Z \to Z$.

This shows that X is of the same homotopy type as Z.

THEOREM:

If x and y a topological spaces, X is path connected and $g: X \to Y$ is a continuous surjective mapping, then Y is path connected.

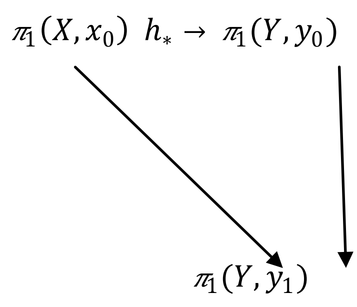
Proof:

Let $a, b \in Y$,then there are point $a^{'}, b^{'} \in X$ such that $g(a^{'}) = a$ and $g(b^{'}) = b$. Now X is path connected, so there is a path f from $a'$and $b^{'}$.

Consider the composite function gf, which is clearly a path from a to b and this show that Y is path connected.

Lemma:

Let $h, k: X \to Y$ be continuous maps; let $(hx_0) = y_0$ and $(kx_0) = y_1$. if h and k are homotopic, there is a path $\alpha \; in \; Y$ from $y_0$to y_1 such that $k_* = \hat{\alpha} o h_*$ indeed, if $H : X \to Y$ is the homotopy between h and k,then $\alpha$is is the path $\alpha(t) = H(x_o, t)$.



Proof :

Let $f: I \to X$ be a loop in X based at x_0.we show that

$$k_*([f]) = \hat{\alpha}(h_*([f]).$$

This equation states that $[kof] = [\hat{\alpha}] * [h\ o\ f] * \hat{\alpha}$, or equivalently, that

$$[\alpha] * [k\ o\ f] = [h\ o\ f] * [\alpha].$$

This is the equation we shall verify.

To begin, consider the loop f_0 and f_1 in the space $X \times I$ given by the equations

$$f_0(s) = (f(s), 0) \quad \text{and} \quad f_1(s) = (f(s), 1).$$

Consider also the path in c in $X \times I$ given by the equation

$$c(t) = (x_0, t).$$

Then $H\ o\ f_0 = hof$ and $Hof_1 = kof$, while Hoc equals the path α.

Let $F: I \times I \to X \times I$ be the map $F(s,t) = (f(s), t)$. consider the following path.

$\beta_0(s) = (s, 0)$ and $\beta_0(s) = (s, 1)$,

$$\gamma_0(t) = (0, t) \text{ and } \gamma_1(t) = (1, t).$$

then $Fo\beta_0 = f_0$ and $Fo\beta_1 = f_1$, while $Fo\gamma_0 = Fo\gamma_1 = c$.

The beoken-line path $\beta_0 * \gamma_1$ and $\gamma_0 * \beta_1$ and path in $I \times I$ from (0,0) to (1,1);

Since $I \times I$ is convex ,there is path homotopy G between them. Then $FoG$is a path homotopy in $X \times I$ between f_0 ∗c and $c * f_1$. And $H\ o(FoG)$ is a path homotopy in y between

$(Hof_o) * (Hof) = (hof) * \alpha$ and

$$(Hoc) * (Hof_1) = \alpha * (kof),$$

CONCLUSION:

This paper concluded that briefly explained about homotopy theory and also its used in the fields are particularly mathematics, biology, science and engineering and etc.

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