Topologies in Fuzzy Locally Convex Spaces

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Abstract: Our purpose is to give an overview of weak topologies in a fuzzy locally convex space. Firstly, we introduce locally convex spaces in fuzzy context. Furthermore fuzzy version of a semi - norm is obtained. Finally, we introduce a weak topology on a fuzzy locally convex space and the weak star topology on its dual as a generalization of usual weak topology. A special attention is also given to some properties of X- topology on X*, called fuzzy weak star topology.

Keywords

Convex fuzzy set, fuzzy vector space, fuzzy topological vector space, fuzzy locally convex space, weak topology, weak - star topology

1. Introduction

The concept of fuzzy vector space and fuzzy topological vector space was developed considerably since the introduction of fuzzy set by Zadeh[]. In this paper, we are devoted to the study of the weak topology on a fuzzy locally convex space.Suppose X is a fuzzy topological vector space with topology | whose dual X* separates points on X. The X* fuzzy topology is of X is called the fuzzy weak topology $|_w$ of X. Let X_w denote X topologized by this weak topology $|_w$. Then X_w is a fuzzy locally convex space whose dual is also X*.

2. Preliminaries

In this section some definitions and properties are reviewed that are needed for the development of the present article. We give a brief account of the developments right from fuzzy convex space and fuzzy topological spaces up to fuzzy topological vector space.

Now we shall define additive operations sum, difference and conjunction on I (X), the class of fuzzy sets

Definition: The *sum* of two fuzzy sets A and B in a set X ,denoted by $A \oplus B$,is a fuzzy set in X defined by $(A \oplus B)(x) = \min(1, A(x) + B(x))$ for, the *difference*, denoted by AΘB, is a fuzzy set in X defined by $(A\Theta B)(x) = \max(0, A(x) - B(x))$, the *conjunction*, denoted by A&B , is a fuzzy set in X defined by $(A \& B)(x) = \max(0, A(x) + B(x) - 1)$ for all $x \in X$

Definition: [4] A g-*fuzzy topology* on a set X is a family τ of fuzzy sets X such that

1 $X \in \tau$ and $\Phi \in \tau$
2 $A \& B \in \tau$ whenever $A, B \in \tau$ and
3 $(\cup_\lambda A_\lambda) \in \tau$ for any subfamily $\{A_\lambda\}_{\lambda \in J}$ in τ

The ordered pair (X, τ), is called a g-*fuzzy topological space* or *gfts*. Members of τ are called g- open fuzzy sets in X. The complement of a g- open fuzzy sets are called g-*closed fuzzy* sets.

Definition: Let X be a non empty ordinary set τ_1, τ_2 be two fuzzy topologies on X. We say τ_1 is coarser than τ_2 or say τ_2 is finer than τ_1, if $\tau_1 \subseteq \tau_2$.

Definition: Let A be a fuzzy set in a g-fuzzy topological space (X, τ). Then the *closure* of A denoted by cl A and *interior* of A denoted by int A are defined by

$$\text{cl } A = \inf\{B : B \supseteq A, B^c \in \tau\}$$

$$\text{int } A = \inf\{B : B \subseteq A, B \in \tau\}$$

Definition: Let (X, τ) and $(Y, \Im)$ be g-fuzzy topological spaces and f a function from X to Y. Then f is said to be g - *fuzzy continuous* if $f^{-1}(B) \in \tau$ for each $B \in \Im$.

Definition: Let X and Y be non-empty sets. Then by $A \cdot B$ we denote the fuzzy set $X \cdot Y$ for which

$$(A \times B)(x, y) = \min\{A(x), B(y)\} \text{ for every } (x, y) \in X \cdot Y$$

For all scalars k and all $x \in X$, (kA) (x) ε A (x)

Definition: A fuzzy set A in X is called a *fuzzy subspace* if (i) A + A □ A; (ii) k A □ A, for every scalar k. A fuzzy set, A on X is a fuzzy subspace if and only if for all x, y □ X and for reals a, b, A(ax + by) ε A(x) э A(y).

Definition: A fuzzy set A □ X is said to be

(a) *Convex* if kA+ (l - k) A □ A, for all k □ [0, l];

(b) *Balanced* if kx □ A for all x □ A and ′k ′ δ 1.

(c) *Absorbing* if for each x in X there is an ∑ > 0 such that kx □ A for k □ (0, ∑). A is absorbing then A (0) = 1. That is, an absorbing set must contain the origin.

Definition: A *fuzzy semi - norm* on X is a fuzzy set A in X which is absolutely convex and absorbing.

Proposition: Let A be a fuzzy set in X. Then, the following assertions are equivalent.

(1) A is convex (balanced).

(2) A {k x + (1- k) y} ε min {A(x), A(y)} for all x, y □ X and all k □ [0, 1];

A (k x) ε A(x) for all k with ′k ′ δ 1

For each d □ [0, 1], the ordinary set A_d = {x □ X: {A(x) ε d} is convex and balanced.

3. Fuzzy locally convex space

A fuzzy topological vector space is a fuzzy vector space that is also a fuzzy topological space such that the linear structure and the topological structure are vitally connected.

Definition: A fuzzy topology τ on a vector space X is said to be a fuzzy vector topology if

(a) the map f: X × X → X defined by (x, y) → x + y is continuous

(b) the map g: K × X → X, (k, x) → k x is continuous when K has the usual topology and X×X, K × X are given the product fuzzy topologies.

Definition: A *fuzzy locally convex space* (FLCS) is a fuzzy topological vector Space whose topology is defined by a family of semi - norms □ such that) p□ □{x: p(x) = 0 = (0).

Clearly, p is a semi-norm.

Here the condition is imposed precisely so that the topology defined by □ is Hausdorff.

Suppose that x, y be two distinct points in X so that x - y ¯0. Then there is a p in □ such that p(x - y) ¯0; let p(x - y) = ∑ > 0.

If U = {z: p(x - z) < ½ ∑} and V = { z: p(y - z) < ½ ∑}, then U) V = ⎞ and U and V are neighbourhoods of x and y respectively.

Proposition: If X is a fuzzy vector space and p is a semi norm, then V = {x: p(x) < 1} is a convex balancing set which is absorbing at each of its points

Proof:

(a) Let x, y □ V ® p (x) < 1 and p (y) < 1.

Then p[k x + (1- k) y] = k p (x) + (1- k) p (y) < k + (1- k) = 1

4 kx + (1- k) y □ V

Or kV+ (l - k) V □ V, for all k □ [0, l]

So V is convex

(b) Let x □ V ® p(x) < 1 ® p (k x) = k p(x) < 1

4 k x □ V for all x □ V and for all ′k ′ δ 1

So V is balanced

c) If a, x □ V, then k x □ V for all x □ V and for all ′k ′ δ 1

4 a + k x □ V for all x □ V and for all ′k ′ δ 1

So V is absorbing at each point

Definition: For a fuzzy LCS X, let X* denote the space of fuzzy continuous linear functional f: X □ F. Then X* is called *dual space* of X. If x*, y* □ X* and 〈 □ F, then (〈x* + y*) (x) = 〈x*(x) + y*(x), x in X, defines an element 〈x* + y* in X*. Thus X* has a natural vector-space structure.

For convenience we will use $\langle x, x^* \rangle$ or $\langle x^*, x \rangle$ to stand for x*(x), x in X and x* in X*

4. Weak fuzzy topology

Now we shall concentrate on the construction of weak fuzzy topologies generated by family of fuzzy semi - norms and some of their properties

Definition: Let X be a fuzzy normed space. For each x* □ X*, define $p_{x^*}(x) = ' x^*(x) '$or$' \langle x, x^* \rangle '$. Then p_{x^*} is a fuzzy semi-norm and if P = {p_{x^*}: x* □ X*}, P makes X into a fuzzy locally convex space. The topology defined on X by these fuzzy semi-norms is called the *weak fuzzy* (*f wk*) *topology* and is often denoted by or ⎧(X, X*).

Definition: Let X be a fuzzy normed space. For each x □ X, define p_x : X* □ [0,□) by $p_{x^*}(x) = ' x^*(x) '$or$' \langle x, x^* \rangle '$. Then p_x is a fuzzy semi-norm and if P = {p_x: x □ X}, P makes X* into a fuzzy locally convex space. The topology defined by these fuzzy semi-norms is called the *weak-star fuzzy* (*or f weak* or f* wk*) *topology* on X*. It is often denoted by ⎧(X, X*).

Theorem: If X is a fuzzy locally convex space and A is a fuzzy convex subset of X, then cl A = f wk – cl A

Proof: If τ is the original fuzzy topology on X, then f wk □ τ, hence cl A □ f wk – clA.
Conversely, if x □ X – clA, then there is an x*□ X*, an 〈 in R, an ∑ > 0 such that

$$\text{Re } \langle a, x^* \rangle \ \delta \ \langle < \langle + \sum \ \delta \ \text{Re } \langle x, x^* \rangle \quad \text{for all a in cl A.}$$

Hence cl A □ B α {y □ X: Re $\langle y, x^* \rangle$ δ 〈 }. But B is f wkclosed since x* is f wk– continuous.
4 f wk - cl A □ B. Since x □ B, x □ f wk- cl A

Corollary: A fuzzy convex subset of X is g- fuzzy closed if and only if it is fuzzy weakly closed.

If X is a complex fuzzy linear space, then the weak topology on X is the same as the weak topology it has if it is considered as a real fuzzy linear space. It is given as the next theorem.

Theorem: Let X be a complex fuzzy locally convex space and let X_R* denote the collection of all fuzzy continuous real linear functions on X. Let the semi-norms on X be defined by using the elements of X_R* and let ⌠ (X, X_R*) be the corresponding topology. Then

⌠(X, X*) = ⌠(X, X_R*)

Proof:

⌠(X, X*) is the topology defined by the family of fuzzy semi-norms {p_{x*}: x* □ X*}, where

p_{x*} (x) = ′ 〈x, x*〉 ′ and

⌠(X, X_R*) is the topology defined by the family of fuzzy semi-norms {p_{x*}: x* □ X*}, where

p_{xR*}(x) = ′ 〈x, x_R*〉 ′

But ′ 〈x, x*〉 ′ and ′ 〈x, x_R*〉 ′ are the same. The result follows.

Proposition: If X is a fuzzy locally convex space. Then f wk is the smallest topology on X such that x* in X* is fuzzy continuous.

Proposition: If X is a fuzzy locally convex space and X* be its duel space. Then f wk* is the smallest topology on X* such that for each x in X, x* □ 〈x, x*〉 is fuzzy continuous.

Definition: If A □ X, the *polar* of A, denoted by A°, the subset of X* defined by

A° = {x* □ X*: ′ 〈a, x*〉 ′δ 1 for all a in A}.

Definition: If B □ X*, the *pre polar* of B, denoted by °B, the subset of X defined by

°B = {x □ X: ′ 〈x, b*〉 ′δ 1 for all b* in B}.

Definition: If A □ X, the *bipolar* of A is the set °(A°). It is also denoted by °A°.

Proposition: If A □ X, then polar of A is a fuzzy subspace of X*

Proof:

A° = {x* □ X*: ′ 〈a, x*〉 ′δ 1 for all a in A} is a fuzzy set in X*.

If x*, y* □ X* and 〈 □ F, then (〈x* + y*) (x) = 〈x*(x) + y*(x), x in X, defines the element 〈x* + y* in X*. (i) A° + A° □ A°; (ii) 〈 A° □ A°, for every scalar〈.

Proposition: If A □ X, then polar of A is convex and balanced

Proof: The polar of A is given by

A° = {x* □ X*: ′ 〈a, x*〉 ′δ 1 for all a in A}.

Let x*, y* □ X*. Then ′ 〈a, x*〉 ′δ 1, ′ 〈a, y*〉 ′δ 1 for all a in A

Now ′ 〈a, k x* + (l - k) y* 〉 ′= k ′ 〈a, x*〉 ′+ (1- k) ′ 〈a, y*〉 ′δ 1 for all k □ [0, l]

k x* + (l - k) y* □ A°

Thus, k A° + (l - k) A° □ A° for all k □ [0, l]

Or A° is convex

Also, ′ 〈a, k x* 〉 ′= k ′ 〈a, x*〉 ′δ 1 for all k □ [0, l]

Thus, k A° □ A°

4 A° is balanced

Conclusion

Here we introduced a brief view on the weak topologies (wk & wk*) on a fuzzy locally convex space. If A , B □ P(X) , then A □ B = A ∗ B , A& B = A) B and AϴB = A \ B. Then the ordinary topology become special case of g- fuzzy topology The weak topology on a complex fuzzy linear space is the same as the weak topology on a real fuzzy linear space. The weak* topologies have very important properties in fuzzy compactness.

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