Periodic function on cyclic groups

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ABSTRACT

This article review on periodic function on cyclic groups.In group theory,a branch of mathematics ,a torsion group or a periodic group is a group in which every element has finite order.All finite groups are periodic.Every cyclic group is abelian.It states thet every finitely generated abelian group is a finite direct product of primary cyclic group.

INTRODUCTION:

In group theory, a branch of abstract algebra , a cyclic group that is generated by a single element.That is, it is a set of invertible elements with a single associative binary operation,and it contain an element g such that every other element of the group may be obtained by repeatedly applying the group operation to g or its inverse.This element g is called a generator of the group.

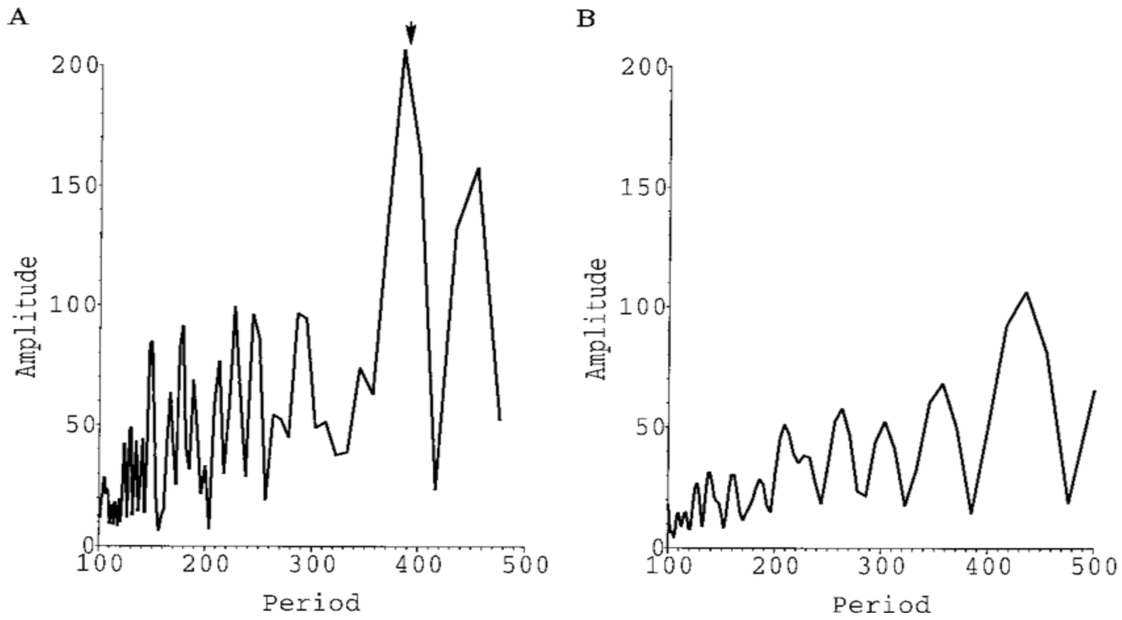
Every cyclic group of prime order is a simple group which cannot be broken down into smaller groups.Every cyclic group is abelian.

KEYWORDS;

Cyclic group,binary operation,abelian group,homomorphism

Definition

A function f:z→c is said to be periodic with period n if F(x)=f(x+n) for all x□ z.



Example:

f:z_n→C is a function then, f=□ x_0,f□ x_0+….+□ x_{n-4},f□ x_{n-1}.

Definition

We call □ a fundamental (primitive) period of f if □ □ □ is the smallest amongst all periods.

Theorem

Given a meromoephic function f define on a region □ (as discussed about).Then there exists a unique meromorphic function F in □ which is the image of □ under □$^{2□□/□}$, such that f(z)=F (□$^{2□□□/□}$).

Proof:

Suppose f is meromorphic in □ in the Z-plane with periodic □ .

f(z)=f(log □

f(z)=F(C)

then clearly F is meromorphic in the z-plane when ever f(z) is meromorphic in the z-plane.

Example

Let o<□ q□ <1. Consider function

f(z)=$\sum^{\infty}$ □2 (−1)$^{□}$ □ 2 □$^{□□□}$

□=∞

which represent a 2□ periodic entire function in c , in fact ,this is a complex form of a fouries series

Let C=□$^{□□}$

F(C)=$\sum^{\infty}$ □2 □$^{□}$

(−1)$^{□}$ □2

□=∞

Which can be show to converge in the function plan 0<□ g□ <+□ ,

Thus, we have

f(z)=F(C)

here we have □ =2π, thus ,the function F is analytic in 0<□ g□ <+□ .

More generally , if the series ,

f(2):=F(□)

$$\sum_{□=\infty}^{\infty} = □□ □\frac{2□□□□}{□},$$

□=□ 2□□□/□

Is a □-periodic analytic function in the infinite horizontal strip {□:□□<f(□)<□□2 }

we can represent the co-efficient

c =1/2πi 1 □(□) d□ r <□ □ □ <r

k ∫□ □ □=0 □□+1 1 2

=1/□∫□+□ (□) □ $\frac{-2□□□□}{□}$ dz,

□

Where a is an arbitrary in the infinite strip {□:□□1 <f(□)<□□1 } and the intergration is take along any path lying in the strip.

Let f be a periodic function of period 2π such that $F(x)=\pi^2-x^2$ for $-\pi<x<\pi$

Solution:

So f is periodic with period 2π and its graph is , We first if f is even or odd.

$$F(-x) =\pi^2-(-x)^2$$

$=\pi^2-x^2$

=f(x)

Since f is even ,

$B_n=0$

A_n =2/π □ (□) cos(□□)□□ $\int_0$

Using the formula for the Fourier coefficient we have ,

$$□□ = 2/\pi \int_0^{□} (□) \cos(nx)\, dx$$

$$= 2/\pi \int_0 (□^2 - □^2) \cos(nx)\, dx$$

$$= 2/\pi([(\pi^2 - x^2) \sin nx/n]_0^{\pi} - \int_0^{□} -2□ \sin nx/n\, dx)$$

$$= 2/\pi([(\pi^2 - \pi^2) \sin n\pi/n - (\pi^2 0) \sin □ /n] + [\int_0^{□} 2□ \sin nx/n dx]$$

$$= 2/\pi\ 2/n \int_0^{□} □ \sin(nx)\, dx$$

$$= 2/\pi\ 2/n([-x \cos nx/n]_0^{\pi} - \int_0^{□} (-\cos nx/n)\, dx)$$

$$= 2/\pi\ 2/n\ 1/n([-\pi \cos n\pi] + [\sin nx/n^2]_0^{\pi}$$

$= \{ -4/n^2$ if n is even, $4/n^2$ if n is odd It remains to calculate a_0

$$a_0 = 2/\pi \int_0 (□^2 - x^2)\, dx$$

$$= 2/\pi[\pi^2 - x^2]_0^{\pi}$$

$$= 4\pi^2/3$$

The Fourier series of f is there fore

$$f(x) = 1/2(a_0 + a_1 \cos x + a_2 \cos 2x +)(b_1 \sin x + b_2 \sin x + \dots\dots\dots)$$
$$= 2\pi^2/3 + 4(\cos x 1/4 \cos 2x + 1/9 \cos x + 1/16 \cos 4x + 1/25 \cos 5x + \dots\dots).$$

The Fourier transform encodes this information as a function.

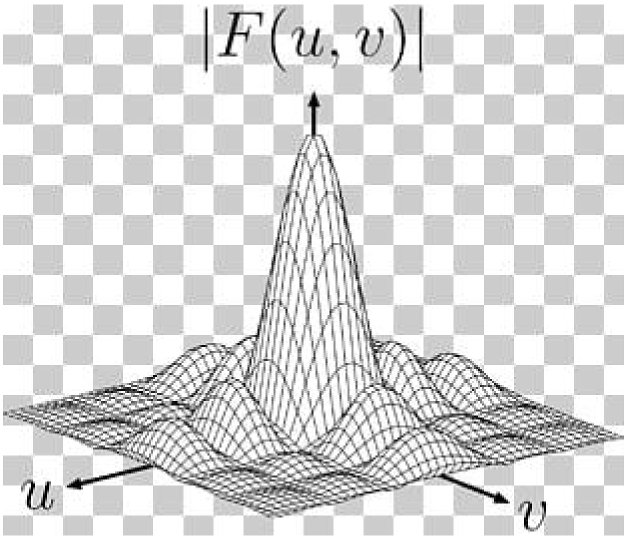
Definition

Let $f{:}z_n \rightarrow c$, define the Fourier transform $f{:}z_n \rightarrow c$ of f by

$$\hat{□}(\bar{□}) = n < x_m, f > = \sum_{□=0}^{□-1} □^{-2□□□□} f(\hat{□})$$

It is immediate that the Fourier transform is a linear transformation

$T:L(Z_n)\rightarrow L[Z_n]$ by the linearity of inner products in the second variable.



proposition

The Fourier transform is invertible more , precisely, $f=1/n\sum_{k=0}^{n-1}\overline{\hat{f}(k)}x_k$

The Fourier transform on cyclic groups is used in signal and image processing. the idea is the values of $\check{f}$ correspond to the wavelengths associated to be wave function f . one sets to zero all sufficiently small values of $\hat{f}$, there by compressing the wave .To recover sometime close enough to the original wave ,as far as our eyes and ears are concerned , one applies Fourier inversion

The Convolution Product

We now introduce the convolution product on L(G) , there by explaining the terminology groups algebra for L(G).

Definition

Let G be a finite group and a,b $\in$ L(G). then the convolution $a*b:G\rightarrow C$ is defined by

$a*b(x)=\sum_{y\in G} a(xy^{-1})\, b(y).$

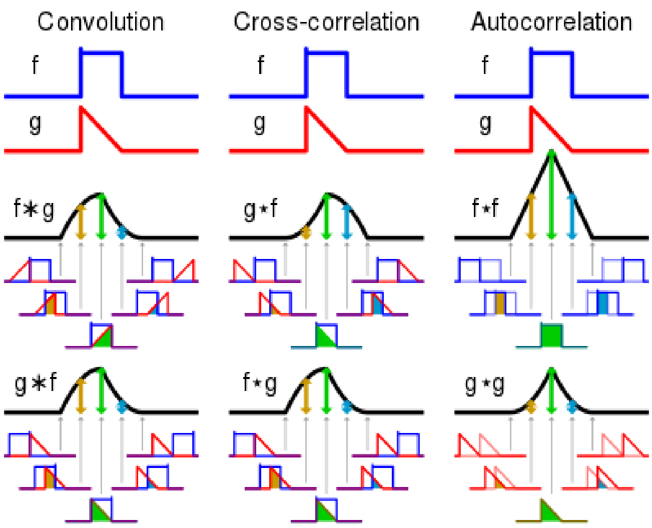
our eventual goal is to show that convolution givens L(G) the structure of a rings. Before that, let us motivate the definition of convolution . To each element g□ G, we have associated the delta function $□_{g.}$ what could be more natural than to try and assign a multiplication * to L(G) so that

Let's show that convolution has this property. Indeed

$□_g*□_h(x)=\sum_{□\in□} □□(xy^{-1})□_h(y)$

And the only non-zero term is when y=h and $g=xy^{-1}=xh^{-1}$, i.e., x=gh. In this case

, one gets 1,so we have proved:



Proposition

For g,h□ G, $□_g*□_h=□_{gh}$.

Now if a,b□ L(G), then

$a=\sum_{□\in□}(□)□_g$, $b=\sum_{□\in□}□(□)□_g$

So if L(G) were really a ring, then the distributives law would yield

$a*b=\sum_{□,h\,□□}□(g)b(h)□_g*□_h$

$=\sum_{□,h\in□}□(□)□(h)□_x$

Applying the change of variables x = gh, y=h then given us $a*b=\sum_{\square\in}(\sum_{\square\in\square}\square(\square\square^{-1})b(y))\square_x$

Theorem

The set L(G) is a ring with addition taken pointwise and convolution as multiplication. More over , $\square_1$ is a multiplicative identity.

Proof:

We will only verify that $\square_1$ is the identity and the associativity of convolution. The remaining verification that L(G) is a ring are straightforward and will be left to reader .

Let a□ L(G).Then

$$a*\square_1(x)=\sum_{\square\in\square}\square(xy^{-1})\square_1(y^{-1})$$
$$=a(x)$$

Since $\square_1(y^{-1})=0$ except when y=1 . Similarly, $\square_1*a=a$. This proves $\square_1$ is the identity .

For associativity, let a,b,c□ L(G). Then

$[(a*b) *c] (x)=\sum_{\square\in}[\square * \square](xy^{-1})\, c(y)$

$$=\sum_{\square\in\square}\sum_{\square\in\square}(xy^{-1}z^{-1})b(z)c(y). \qquad \rightarrow(*)$$ We

make the change of variables u=zy (and so $y^{-1}z^{-1}=u^{-1}$,$z=uy^{-1}$).

The right hand of (*)

$$\sum_{\square\in\square}\sum_{\square\in\square}(xu^{-1})b(uy^{-1})c(y)=\sum_{\square\in\square}\square(xu^{-1})\sum_{\square\in\square}\square(uy^{-1})c(y)$$

$=\sum_{\square\in\square}(xu^{-1})[b*c](u)$

$=[a*(b*c)](x)$

Completing the proof.

Proposition

Z(L(G)) is the canter of L(G).That is , f:G→C is a class function if and only if a*f=f*a for all a□ L(G).

Proof:

Suppose first f is a class function and let a□L(G).Then $a*f(x)=\sum_{\square\in\square}\square(xy^{-1})f(y)$

$=\sum_{\square\in\square}(xy^{-1})f(xy^{-1}x)$ →(*)

Since f is a class function . setting $z=xy^{-1}$ turns the right hand side of (*) into

$\sum_{\square\in\square}(z)f(xz^{-1}) = \sum_{\square\in\square}\square(xz^{-1})a(z)$

$= f*a(x)$

And hence a*f=f*a.

For the other direction, let f be in the canter of L(G).

Claim. f(gh) for all g,h□ G

Proof of claim:
Observe that

$f(gh)=\sum_{\square\square\square}\square(gy^{-1})\square_{h-1}(y)$
$=f*\square_{h-1}(g)$
$=\square_{h-1}*f(g)$
$=\sum_{\square\in\square}\square h-1(gy^{-1})f(y)$
$=f(hg)$

Since $\square_{h-1}(gy^{-1})$ is non-zero if and only if $gy^{-1}=h^{-1}$, that is ,y=hg.
Complete the proof.

Conclusion

A cyclic group is a group with an element that has an opertion applied that produces the whole set.A cyclic group is the simplest group.A cyclic group could be a pattern found in nature for example in a snowflake ,number theory,and in pure mathematics

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