INVARIENTS FROM THE SEIFERT MATRIX

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ABSTRACT

For a mathematician it is natural to *ask*, since we have such a nice tool as the Seifert matrix, what matrix properties do we know that, via A_1 *and* A_2, might yield a knot (or link) invariant.

INTRODUCTION

Mathematical studies of knots began in the 19^{th}century with Carl Friedrich Gauss, who defined the linking integral (silver 2006). In the 1860s, Lord Kelvin's theory that atoms were knots in the aether led to Peter Guthrie Tait's creation of the first knot tables for complete classification.

While tabulation remains an important task, today's researchers have a wide variety of background and goal.

In the last 30 years knot theory has also become a tool in applied mathematics. Chemists and biologists use knot theory to understand.
For example, chirality of molecules and the actions of enzymes on DNA.

KEYWORDS

Matrix , Alexander polynomial, symmetric polynomial, Seifert matrix.

The Alexander polynomial:

For a mathematician it's natural to ask, since we've such a pleasant tool because the Seifert matrix, what matrix properties can we know that, via
$A1$ *and* $A2$, might yield a knot (or link) invariant.

Exercise

Find an example that shows that the determinant, det

M, of the Seifert matrix M of a knot K is *not* a knot invariant.

Proof:

However, we should not discard the idea of using the determinant. Let us, first, symmetrize the matrix M to form the matrix sum $M + MT$
If we now check out absolutely the value of the determinant of $M + MT$, this does cause a link invariant.

Proposition

If M is the Seifert matrix of knot (or link)

K, then $|\det(M + M^T)|$

is an invariant of the knot K.

This invariant is named the determinant Of K.

The Alexander-Conway polynomial:

The reader will soon find, by experimenting with the above procedure, that. if we wish to use the Alexander polynomial to get a minimum of a partial knot table, the above procedure is sort of cumbersome

However, due to the constant state of flux in knot theory and its interaction with other disciplines, the above problem can be obviated.

In the late 1950s and 1960s, computers were transformed from a research project into a research tool.

Although the number-crunching abilities of computers were of tremendous advantage, an extra impetus was still required to make the Alexander polynomial more computer friendly.

This spark of ingenuity was provided by J.H. Conway within the late 1960s, when he devised a particularly efficient mechanical procedure to compute the Alexanderpolynomial.

(With hindsight, if we carefully reread Alexander's original paper, it is possible to glean from it Conway's method.

So perhaps, rather like in the case of fractals, this is a case of technology catching up with mathematical theory).

Exercise

Show that if $K_+$is a μ-component link, then K_- is also a μ-component link, but $K_0$is either a (μ-1)-component or a (μ + 1)-component link.

Proof:

The polynomial (Z), defined as above, is called the *Conway polynomial.* To actually show that the Laurent polynomial (Z) ,

obtained from Axioms 1 and 2, is well-defined and unique is quite troublesome (a complete proof can be found in [LM]).

However, if we assume the well-definedness anduniqueness of (Z) , then by proving the following theorem,

we can show that V_K(z) and the Alexander polynomial are essentially the same.

Theorem

Suppose that f(t) may be a Laurent polynomial that satisfies the subsequent two conditions:

1) $(1) = 1$

2) $(t) = (t^{-1})$

Then there exists a knot that has as its Alexander polynomial f(t). Equivalently, if g(z) is an integer polynomial in Z^2 with g(O) = 1,

then there exists a knot K that has as its Alexander-Conway polynomial g(z).

The proof requires finding an appropriate orientable surface, F, with its Seifert matrix M of order k satisfying

$$\boldsymbol{t^{-k/2}\det(M - tM^T) = f(t).}$$

Theorem

Suppose A is a nxn symmetric matrix with its entries real numbers. Then it is possible to find a real (with its entries real numbers) invertible matrix P such that $PApT$= B is a diagonal matrix.

In ., addition, we may assume that

det P = ±1.

Proof:

We may rephrase the essence of this theorem in the terminology.

A symmetric matrix is $A1$-equivalent to a square matrix .

which may be a bit tedious and can not shed any insight in what follows, we propose for instance the tactic of diagonalizing a matrix by means of a few of examples.

These examples will, hopefully, indicate to the reader the idea of the proof, and thus the proof will become only an exercise in (the manipulations of) linear algebra.

Theorem

Suppose K is a knot, then $\Delta(t)$ is a symmetric Laurent polynomial ie)

$\Delta(t) = a_{-n}t^{-n} + a_{-(n-1)}t^{-(n-1)} + \cdots + a_{n-1}t^{n-1} + a_n t^n$

and ,

$a_{-n} = a_n, a_{-(n-1)} = a_{n-1}, \ldots, a_{-1} = a_1.$

Proof:

Suppose that M is a seifert matrix of K and k is the order of M. Since K is a knot, k is necessary even. Therefore

$$\Delta_k t^{-1} = t^{k/2}\det(M - t^{-1}M^T) = t^{-k/2}\det(tM - M^T)$$

$$= (-1)^{-K/2}\det(M^T - tM)$$

$= t^{-k/2}\det(M - tM^T)^T$

$= t^{-k/2}\det(\mathrm{M} - tM^T)$

$= \Delta(\mathrm{t})$.

Preposition

$|\Delta(-1)|$ is equal to the determinant of a knot K.

Proof:

$|\Delta(-1)| = \quad (-1)^{-K/2}det(M + M^T)$

$$= |\det(M + M^T)|$$

Theorem:

If K is a knot, then n(k)=0 and σ(k) is always even.

Proof:

The seifert matrix, M for K is a square matrix of even order.

Since det($M - M^T$)= Δ_K(1)= 1

det($M + M^T$) is an odd integer and so non-zero

consequently, n($M + M^T$)= 0

and n(k)= 0

Therefore the number of eigen values of ($M + M^T$) that are not zero is even;

Hence σ($M + M^T$) is also even.

Theorem:

Suppose that M_1 and M_2 are the seifert matrices for a knot (or link) K. Further, if r and s are respectively, the orders of M_1 and M_2 then the following equality holds.

$$t^{-r/2}\det(M1 - tM1^T) = t^{-s/2}\det(M2 - tM2^T)$$

Proof:

Therefore if M is a seifert matrix of K and its order is K, then

$$t^{-r/2}\det(M - tM^T)$$

Is an invariant of K. This invariant is known as the Alexander polynomial of K and is denoted by $\Delta(t)$.It follows directly from our previous discussions that k=2g(F)- μ(k)-1, Where as before F is the seifert surface from which we have constructed M, and μ(k) is the number of components of the link K.

In most cases Δ(t) has some terms with a negative exponent;

However, if we multiply Δ(t) by a suitable factor then we can obtain a polynomial with only positive exponents.

Sometimes it is preferable to work with such an interpretation of Δ(t).If K is a link with an even number of components, then K is odd.

Therefore for such links Δ(t) is a polynomial with terms as powers of

$t^{-1/2} = \sqrt{t}$ or ($t^{-1/2} = 1/\sqrt{t}$).In these cases we define $\boldsymbol{(t^{1/2})^2 = t}$.

Theorem:

Suppose $K_1 \neq K_2$ is the connected sum of two knots (or links) K_1 and K_2 then

$\Delta \boldsymbol{K}_1 \neq \boldsymbol{K}_2$**(t)**$=\Delta \boldsymbol{K}_1 \Delta \boldsymbol{K}_2$

Proof:

Firstly, create in the prescribed way the seifert surfaces F_1 and F_2 of respectively, K_1 and K_2.

Then the orientable surface formed by joining these surfaces by a band becomes a seifert surface for $K_1 \neq K_2$.

If we suppose M_1 and M_2 are the seifert matrices of K_1 and K_2 obtained from F_1 and F_2, then M the seifert matrix of $K_1 \neq K_2$ has following form,

$$M = \begin{bmatrix} M1 & 0 \\ 0 & M2 \end{bmatrix}$$

Therefore,

$$\det(M - tM^T) = \det(M_1 - tM_1^T)\ \det(M_2 - tM_2^T)$$

If L is μ-component link, then we may write $\nabla_L(Z) = Z^{\mu-1}$ g(z),where g(z) is an integer polynomial in z^2.

So if we let,

$$\tilde{\Delta}(t) = g(\sqrt{t} - 1/\sqrt{t})$$

Then,

$$\tilde{\Delta}(t^{-1}) = \tilde{\Delta}(t)$$

Andthus $\tilde{\Delta}(t)$ isasymmetricintegerpolynomial,thispolynomialis called the **hosokawa polynomial**.

Theorem:

Any two s-equivalent non singular matrices can be joined by a sequence of the following two types of moves.

1) right enlargement, then leftreduction.

2) left enlargement, then right reduction. Moreover we can do all of type(i) first and then all oftype(ii).

Proof :

Thus we never have to deal with matrices much larger than the original one.

The next step would be to examine a singular move of the type(i) or type(ii) and be able to write down all the matrices obtained by such a move from a given one.

A priori this may seem improbable. since the vectors and used in the enlargement may vary over an infinite number of choices.

But, in fact only a finite number of distinct (up to congruence) enlargements occur and these can be constructed in a finite number of steps.

Suppose A has rank r, consider the free abelian group of rank r, written as columns of A.

Let the quotient group be denoted V(A). Then V(A) is a finite group with det A elements.

Let O(A)= $\{\boldsymbol{P\ unimodular} : \boldsymbol{PAP' = A}\}$

Be the orthogonal group of A. Then O(A) acts on V(A) by left multiplication.

Given A clearly one can completely write down this situation.

Corollary

Two unimodular matrices are S-equivalent if and only if they are already congruent.

This follows immediately from the above results and $(A) = 0$.

For example fibered knots have unimodular seifert

matrices. As illustration I would like to give

some example..

CONCLUSION

This provides a bridge between knot theory and graph theory. The study of Reidemeister moves, some classical invariants like crossing number, knotting number, bridge number and other invariants like the genus of a knot and some polynomial invariants have been discussed here.

The survey has shown that the fundamental problem of knot theory was the process of distinguishing knots. Many invariants have been discovered to show that two knots are not equivalent. If an invariant of two knots is equal it did not necessarily imply that the knots are equivalent.

This necessitated further research o Also classifying knots and studying them with a topological point of view is becoming essential in the inter disciplinary field as well leading to a great scope to explore thefield.

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