# DECOMPOSITION OF FUZZY MATRIX

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## ABSTRACT

This paper has two chapters, in chapter one, basic concepts about fuzzy matrices are introduced. Basic notations of matrices are given in section one in order to make the book self-contained. Section two gives the properties of fuzzy matrices. Since the data need to be transformed into fuzzy models, some elementary properties of graphs are given. Further, this section provides details of how to prepare in linguistic question to make use of in these fuzzy models when the data related with the problem is unsupervised.

## INTRODUCTION

This paper aims to assist social scientists to analyze their problems using fuzzy models. The basic and essential fuzzy matrix theory is given. The paper does not promise to give the complete properties of basic fuzzy theory or basic fuzzy matrices. Instead, the authors have only tried to give those essential basically needed to develop, the fuzzy model. The authors do not present elaborate mathematical theories to work with fuzzy matrices. Instead they have given only the needed properties by way of examples. The authors feel that the paper should mainly help social scientists, who are interested in finding out ways to emancipate the society. Everything is kept at simplest level and even difficult definitions, have been omitted. Another main feature of this paper is the description of each fuzzy model using examples from real-word problems. Further. This paper gives lots of reference so that the interested reader can make use of them.

This paper has two chapters, in chapter one, basic concepts about fuzzy matrices are introduced. Basic notations of matrices are given in section one in order to make the book self-contained. Section two gives the properties of fuzzy matrices. Since the data need to be transformed into fuzzy models, some elementary properties of graphs are given. Further, this section provides details of how to prepare in linguistic question to make use of in these fuzzy models when the data related with the problem is unsupervised.

## DEFINITION

Some operations and notation are defined, For x, y in the interval [0, 1], x+y, xy,x – y, x*y are defined as follows.

x +y = max(x, y)

xy = min(x, y)

$$x - y = \begin{cases} x \ \text{if} \ x > y \\ 0 \ \text{if} \ x \leq y \end{cases}$$

$$x * y = \begin{cases} 1 \ \text{if} \ x \geq y \\ x \ \text{if} \ x < y \end{cases}$$

Next, we define some matrix operations on fuzzy matrices whose elements exists in the interval [0, 1].

Let

$$A = [a_{ij}] \ (m \times n)$$
$$B = [b_{ij}] \ (m \times n) \ \cdot$$

$F = [\ f_{ij}\ ], \quad (n \times l)$, and $R = (r_{ij}), \quad (n \times n)$

Then the following operations are defined.

$$AF = \left[ \sum_{k=1}^{n} a_{ik} f_{kj} \right]$$

$$A * F = \left[ \prod_{k=1}^{n} (a_{ik} * f_{kj}) \right]$$

$A' = [a_{ij}]$ (The transpose of A)

$A \leq B$ If and only if $a_{ij} < b_{ij}$ for every i, j. $\Delta R = R - R'$.

**TRANSITIVE:**

A matrix R is said to be transitive if $R^2 \leq R$

**REFLEXIVE:**

A matrix R all of whose diagonal elements are one is called reflexive.

**IRREFLEXIVE:**

Conversely a matrix R all of whose diagonal elements are zero is called irreflexive.

**NILPOTENT:**

A matrix R is nilpotent if $R^n = 0$ (0 is the zero matrixes) we deal only with fuzzy matrices.

## LEMMA:

If $A = [a_{ij}]$ is a $m \times n$ fuzzy matrix then $A * A'$ is reflexive and transitive.

**PROOF:**

Let $S = [S_{ij}] = A * A'$.

$(ie)\quad S_{ij} = \prod_{k=1}^{n} (a_{ik} * a_{jk}) = 1$

Clearly,

$$S_{ii} = \prod_{k=1}^{n} (a_{ik} * a_{jk}) = 1.$$

Thus, S is reflexive.

Suppose that

$S_{il} S_{lj} = c > 0$ For some l, then

$$S_{il} = \prod_{k=1}^{n} (a_{ik} * a_{lk}) \geq c$$
$$S_{ij} = \prod_{k=1}^{n} (a_{ik} * a_{jk}) \geq c.$$

If $S_{ij} < c$ then $a_{ih} < a_{jh}$ and $a_{ih} < c$ for some h.

Therefore $\because S_{il} \geq c$

And $S_{lj} \geq c$ we have $c > a_{ih} \geq a_{lh} \geq a_{jh},$

Which is the contradiction

Hence $S_{ij} \geq c$ so that S is transitive.

Let $A_i$ be the $i^{th}$ row of A if $A_i \geq A_j$ then $S_{ij} = 1$ where $S_{ij}$ is the (i, j) entry of $S = A * A'$.

Hence the matrix S represents inclusion among the rows of A.

In other word S gives the hierarchy is reflexive and transitive.

This becomes clear if A is Boolean.

Interesting properties

If a $n \times n$ fuzzy matrix R is reflexive and transitive then as is well-known R is idempotent that is

$$R^2 = R \; .$$

## LEMMA:

Let $S = [S_{ij}]$ and $Q = [q_{ij}]$ be $m \times m$ transitive matrices. If $S \leq Q$ then $S - Q'$ is reflexive and transitive.

## PROOF:

Let $H = [h_{ij}] = S - Q'$ that is $h_{ij} = S_{ij} - q_{ji}$ then $h_{ii} = S_{ii} - q_{ii} = 0$ .

So that H is irreflexive next suppose that $h_{ik} h_{kj} = c > 0$ then there are two cases.

Case (i)

$S_{ik} = C$ , $S_{ik} > Q_{ki}$ , $S_{kj} \geq C$

Case (ii)

$S_{ik} \geq C$, $S_{kj} = C$ , $S_{kj} > q_{jk}$ Clearly $S_{ij} \geq C$ suppose that $q_{ij} \geq c$ in the first cases $q_{jk} \geq q_{ji} q_{ik} \geq c$ .

This is contradiction.

Hence $q_{ji} < c$ so that $h_{ij} \geq c$

That is H is transitive.

Hence the proof

# ON FUZZY M-NORMED MATRICES

## PRELLIMINARIES:

We shall consider F fuzzy algebra [0, 1] with operations (+,*) and standard order ≤ where a+b=max {a, b}, a.b=min {a, b} for all a, b in F. F is a commutative semi ring with additive and multiplicative identities 0 and 1, respectively.

Let $M_{mm}(F)$ denotes the set of all $m \times n$ fuzzy matrices over F. In short $M_n(F)$ is the set of all fuzzy matrices of order n.

Define '+' and scalar multiplication in $M_n(F)$ as $A + B = [a_{ij} + b_{ij}]$ where $A = [a_{ij}], B = [b_{ij}]$ and $CA = [Ca_{ij}]$ where $C \in [0,1]$ with these operations $M_n(F)$ forms a vector space over [0, 1]

In all vector space more properties can be analyzed if the vector spaces and supplied with matrices. The matrices are defined in vector spaces through the introduction of suitable non-negative quantity called norm. In $M_n(F)$ also the same technique is adopted by introducing the concept norm in the following way.

## FUZZY M-NORMED AND SEMIMETRIC

## DEFINITION

Let $M_n(F)$ be the set of all $(n \times n)$ fuzzy matrices overF =[0,1].

For every A in $M_n(F)$ define m-norm of A denoted by $\| A \|_m$ as $\| A \|_m = \max[a_{ij}]$ where $A = [a_{ij}]$

(or) $= \max[a_{11}, a_{12}, \ldots\ldots\ldots a_{ij}, \ldots\ldots\ldots a_{mn}]$

(Or)

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}..$$

## THEOREM:

If $M_n(F)$ is the set of all $(n \times n)$ fuzzy matrices over F=[0,1] then for all fuzzy matrices A and B in $M_n(F)$ and any scalar C in [0,1] We have,

i) $\| A \|_m > 0$ $and \| A \|_m = 0$ $iff$ $if$ $A = 0$

ii) $\| CA \|_m = C \| A \|_m$ for any C in [0, 1].

iii)i $\| A + B \|_m = \| A \|_m + \| B \|_m$ for A, B in $M_n(F)$ .

v) $\| AB \|_m = \| A \|_m \| B \|_m$ for A, B in $M_n(F)$ .

## PROOF:

Let A=$[a_{ij}]$ and B=$[b_{ij}]$ be two fuzzy matrices.

(i) since all $a_{ij} \in [0,1]$ $\max[a_{ij}] = \|A\|_m \geq 0$ for all $A \in M_n(F)$

$\Rightarrow a_{ij} = 0$ For all i and j

$\Rightarrow$ A=0

Contradiction if A=0 then $\max[a_{ij}] = 0$

$\Rightarrow \|A\|_m = 0$

$\therefore \|A\|_m = 0$ Iff if A=0

(ii) If $C \in [0,1]$ Then CA=$[Ca_{ij}]$

$$\therefore \begin{aligned} \|CA\|_m &= \max[Ca_{ij}] = C \max[a_{ij}] \\ &= C\|A\|_m \end{aligned}$$

(iii)(iii) $\|A\|_m$=max $[a_{ij}]$ and $\|B\|_m$ =max $[a_{ij}]$

Now $\|A+B\|_m$ =max$[C_{ij}]$ where $[C_{ij}] = [a_{ij}]$

=max $\{[a_{ij}] + [b_{ij}]\}$

=max$[a_{ij}]$ +max$[b_{ij}]$

$= \|A\|_m + \|B\|_m$ .

(iv)(iv) $\|A\|_m$ =max$[a_{ij}] = a_{ij}$ and $\|B\|_m$ =max [ $b_{ij}$ ]= $b_{ij}$

If AB=D then the entries of D are given by,

$$d_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$$= \sum_{k=1}^{n} \{\min(a_{ik}, b_{kj})\}$$

$$=\min\left((a_{i1}, b_{1j}) + \min(a_{i2}, b_{2j}) + \dots\dots\dots + \min(a_{in}, b_{nj})\right) \quad \dots\dots\dots\dots (1)$$

Case (i)

If all $a_{ij} \leq [b_{ij}]$ for j=1, 2…. n

Then we have $d_{ij} = a_{i1} + a_{i2} + \dots + a_{in}$ (from (1))

$= a_{ij}$

$\therefore \max[d_{ij}] = \max[a_{ij}]$

(i.e.) $\|AB\|_m = \|A\|_m = \|A\|_m \|B\|_m$

Case (ii):

If all $b_{ij} \leq a_{ij}$, for j=1, 2…. n

Then we have $d_{ij} = b_{i1} + b_{i2} + \dots + b_{in}$ $(\,from\ (1)\,)$

$= b_{ij}$

$\therefore \max[d_{ij}] = \max[b_{ij}]$

(i.e.) $\|AB\|_m = \|B\|_m = \|A\|_m \|B\|_m$

Case (iii)

Let some $a_{ij} \leq [b_{ij}]$ and some other $b_{ij} \leq a_{ij}$ without loss of generality. Let we assume that $a_{im} < b_{im}$ for all n<m and $b_{im} < a_{im}$ for all $n \geq m$

$\therefore$ From (1) $d_{ij} = a_{ij} + \dots\dots\dots a_{im} + b_{i(m+1)} + \dots + b_{in}$

$$d_{ij} = \sum_{j=1}^{m} a_{ij} + \sum_{j=m+1}^{n} b_{ij} = a_{ij} + b_{ij}$$

$$d_{ij} = a_{ij}, \text{ if } a_{ij} \geq b_{ij}$$
$$= b_{ij}, \text{ if } a_{ij} \leq b_{ij}$$

$$\therefore \text{Max } [[d_{ij}] = \max[a_{ij}] = \|A\|_m$$

(Or)

$$\text{Max}[b_{ij}] = \|B\|_m$$

$$\|AB\|_m = \|A\|_m \|B\|_m .$$

## EXAMPLE:

If A= $\begin{pmatrix} .7 & .2 & 0 \\ .8 & .3 & .4 \\ .5 & .6 & .9 \end{pmatrix}$ & B= $\begin{pmatrix} .2 & 1 & 0 \\ .5 & .4 & .6 \\ .7 & .5 & .6 \end{pmatrix}$

Then $\|A\|_m = 0.9, \|B\|_m = 0.7$

A+B= $\begin{pmatrix} .7 & .2 & 0 \\ .8 & .4 & .6 \\ .7 & .6 & .9 \end{pmatrix}$

$\|A + B\|_m = 0.9$, $\|A\|_m + \|B\|_m = 0.9 + 0.7 = 0.9$

$\|A + B\|_m = \|A\|_m + \|B\|_m$

Set C=0.2

Then

$$CA = 0.2\begin{pmatrix} .7 & .2 & 0 \\ .8 & .3 & .4 \\ .5 & .6 & .9 \end{pmatrix} = \begin{pmatrix} .2 & .2 & 0 \\ .2 & .2 & .2 \\ .2 & .2 & .2 \end{pmatrix}$$

$$\|CA\|_m = .2,\ C\|A\|_m = (.2)(.9) = .2$$

$$\therefore \|CA\|_m = C\|A\|_m$$

$$AB = \begin{pmatrix} .7 & .2 & 0 \\ .8 & .3 & .4 \\ .5 & .6 & .9 \end{pmatrix}\begin{pmatrix} .2 & .1 & 0 \\ .5 & .4 & .6 \\ .7 & .5 & .6 \end{pmatrix} = \begin{pmatrix} .2 & .2 & .2 \\ .4 & .4 & .4 \\ .7 & .5 & .6 \end{pmatrix}$$

$$\|AB\|_m = 0.7,\ \|A\|_m \|B\|_m = (0.9)(0.7) = 0.7$$

$$\therefore \|AB\|_m = \|A\|_m \|B\|_m . \ \square$$

## DEFINITION:

A fuzzy matrix A is defined to be greater than B if $\|B\|_m \leq \|A\|_m$. A is strictly greater then B if $\|B\|_m < \|A\|_m$ we also say that B is smaller than (strictly smaller) A.

## EXAMPLE:

Let $A = \begin{pmatrix} .2 & .3 & .5 \\ .1 & .5 & .4 \\ 0 & .6 & .7 \end{pmatrix}$ and $B = \begin{pmatrix} .1 & .5 & .4 \\ .3 & .2 & .1 \\ 0 & .5 & .4 \end{pmatrix}$

Then $\|A\|_m = 0.7$, $\|B\|_m = 0.5$

$$\therefore \|B\|_m < \|A\|_m$$

$\therefore$ A is strictly greater then B.

## DEFINITION:

Define a mapping d: $M_n$ (F)$\times$ $M_n$ (F)$\rightarrow$[0,1] as d(A,B)= $\|A-B\|_m$ for all A,B$\in$ $M_n$ (F).

## THEOREM:

The above mapping d satisfies the following conditions for all A, B, C in $M_n$(F).

i) d(A,B) $\geq$ 0, if d(A,B)=0 then A=B

ii) d(A,B)=d(B,A)

iii) d(A,B) $\leq$ d(A,C)+d(B,C) for all A,B,C

$M_n$(F) thus, d is a pseudo-metric in $M_n$(F).

## PROOF:

i) d(A,B)= $\|A+B\|_m \geq 0\ \forall$ A,B in $M_n$(F)

$\therefore$d(A,B) $\geq$ 0

Suppose d (A, B) =0 then $\|A+B\|_m = 0$

$\Rightarrow \|A\|_m + \|B\|_m = 0$

$\Rightarrow$ A=0 and B=0

$\Rightarrow$ A=B

But A=B $\Rightarrow \|A\|_m = \|B\|_m$

(i.e.) $\|A\|_m + \|B\|_m = \|B\|_m + \|B\|_m = \|B\|_m$

$\Rightarrow \|A+B\|_m = \|B\|_m$

$\Rightarrow$ d (A,B) $\neq 0$

$\therefore$ A=B need not imply d (A, B) =0

(i) d (A,B)= $\|A+B\|_m = \|B+A\|_m =$ d(B,A)

$\therefore$ d (A,B)=d(B,A)

(ii) Let A,B,C in $M_n$ (F) be such that $\|C\|_m \geq \|B\|_m \geq \|A\|_m$

d (A,B)= $\|A+B\|_m = \|A\|_m + \|B\|_m = \|B\|_m$

d(A,C)= $\|A+C\|_m = \|A\|_m + \|C\|_m = \|C\|_m$

d(B,C)= $\|B+C\|_m = \|B\|_m + \|C\|_m = \|C\|_m$

d(A,B)= $\|B\| \leq \|C\|_m$ =d(A,C)+d(B,C)

$\therefore$ d(A,B) $\leq$ d(A,C)+d(B,C)

Similarly,for the other cases also we have

d (A,B) $\leq$ d(A,C)+d(B,C)

Thus, in all cases,

d (A,B) $\leq$ d(B,C)+d(C,A) for all A,B,C in $M_n$(F)

Thus from (i) (ii) and (iii) we see that d is a pseudo-metric on $M_n$(F)

The above pseudo=metric can be extended to a finite product of $M_n$(F).

## NOTATION

Let X= $M_n$(F) $\times$ $M_n$(F) $\times$ …….n times.

## THEOREM

The mapping d: X$\times$ X$\rightarrow$[0, 1] defined as $d(\bar{A},\bar{B}) = \sum_{I=1}^{n} d_i (A_i, B_i)$

, where $\bar{A} = A_1 A_2, .........A_n$ and $\bar{B}$ = ( $B_1, B_2, .......B_n$) are in V and $d_i$ are pseudo-metric on $M_n$ (F) is a pseudo-metric for X.

## PROOF:

(i) Since $d_i$ 's are pseudo-matrices

$d_i (A_i, B_i) \geq 0$, i=1, 2,…..n

$\therefore \sum d_i (A_i, B_i) \geq 0$

(i.e.) $d(\bar{A},\bar{B}) \geq 0 \quad \forall \bar{A}, \bar{B} \in$ X

(ii) If $d(\bar{A},\bar{B})$ =0 then $\sum_{i=1}^{n} d_i (A_i, B_i) = 0$

$\Rightarrow$ Max $d_i (A_i, B_i) = 0$

$\Rightarrow d_i (A_i, B_i) = 0$, for i=1, 2,….n

$(\because (A_i, B_i) \in [0,1] \forall$ i)

$\Rightarrow \| A_i + B_i \|_m = 0$

$\Rightarrow \| A_i \|_m + \| B_i \|_m = 0$

$A_i = 0$ and $B_i = 0$ for i=1,2,…n

$$\therefore (A_1, A_2, \ldots\ldots A_n) = (B_1, B_2, \ldots\ldots B_n)$$

(i.e.) $\overline{A} = \overline{B}$

Thus $d(\overline{A}, \overline{B}) = 0 \quad \Rightarrow \overline{A} = \overline{B}$

Conversely if $\overline{A} = \overline{B}$ then $A_i = B_i$, for i=1,2 ,…n

$A_i = B_i \Rightarrow \| A_i \|_m = \| B_i \|_m$ for i=1, 2,…..n

(i.e.) $\| A_i \|_m + \| B_i \|_m = \| B_i \|_m + \| B_i \|_m$ for i=1,2,…..n

(i.e.) $\| A_i + B_i \|_m = \| B_i \|_m$ for i=1, 2,… ..n

(i.e.) $d_i(A_i, B_i) = \| B_i \|_m \geq 0$

$\Rightarrow \sum_{I=1}^{n} d_i(A_i, B_i)$ Need not be equal to zero

(i.e.) $d(\overline{A}, \overline{B})$ need not be zero

$\therefore \overline{A} = \overline{B} \neq d(\overline{A}, \overline{B}) = 0.$

(iii) $d(\overline{A}, \overline{B}) = \sum_{i=1}^{n} d_i \overline{(A_i, B_i)} = \sum_{I=1}^{n} d_i \overline{(B_i, A_i)} = d(\overline{B}, \overline{A})$

(iv) For each $d_i$, i=1,2,…..n

$d_i(A_i, B_i) \leq d_i(A_i, B_i) + d_i(C_i, B_i)$

$$\sum_{i=1}^{n} d_i(A_i, B_i) \leq \sum_{i=1}^{n} d_i(A_i, C_i) + \sum_{i=1}^{n} d_i(C_i, B_i)$$

(i.e.) $d(\overline{A}, \overline{B}) < d(\overline{A}, C) + \overline{d}(\overline{C}, B)$

Thus from (i),(ii), (iii) and (iv) we see that d is pseudo-metric.

# CONCLUSION

In this see we give some basic matrix theory essential to make the book a self-contained one. However, the book of Paul. Horst on matrix algebra for social scientists would be a boon to social scientists who wish to make use of matrix theory in their analysis.

We give some very basic matrix algebra. This is need for the development of fuzzy matrix theory and the psychological problems.

However, these fuzzy models have been used by applied mathematicians, to study social and psychological problems. These models are very much used by doctors, engineers, scientists, industrialists and statisticians. Here we proceed on to give some basic properties of matrix theory.

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