A STUDY ON SPECTRAL SEQUENCE

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ABSTRACT: -

This paper tries to be an introductory text to spectral sequences, a useful technique in algebraic topology which has been frequently used in order to compute homology and homology group of spaces by means of some examples we show the utility of this construction and some of the problem thatdo often appear when dealing with spectral sequences.

KEY WORDS: -

Bigraded object, filtered differential object, associated graded object, Bi - degree, monomorphism.

INTRODUCTION: -

Algebra (from Arabic : al –jabr meaning " reunion of broken parts and bonesetting) is one of the broad parts of mathematics ,together with number theory,geometryandanalysis.initsmostgeneralform,algebraisthestudyof mathematical symbols and the rules for manipulating these symbols ,it is a unifying thread of almost all ofmathematics.

SOME BASIC DEFINITION AND EXAMPLES

DEFINITION: -

$\mathcal{C}$ be a category, A graded object in $X^{\bullet}$ in $\mathcal{C}$ is a $\mathbb{Z}$ - indexed family $\{X^p\}_{p\epsilon z}$ of object in $\mathcal{C}$.

A**bigradedobject** $E^{\bullet,\bullet}$ is a $(\mathbb{Z}, \mathbb{Z})$ indexed family $\{E^{p,q}\}_{(p,q)\epsilon zxz}$ of objects in $\mathcal{C}$.

DEFINITION: -

A spectral sequence is a sequence $\{ , d_r\}$ $(r \geqq 0)$ of graded objects

$$E_r = \bigoplus_{p\geqq 0} E_r^p$$

together with homomorphisms (also called differentials) of degree r,

$$d_r : E_r^p \to E_r^{p+r}$$

satisfying $d^2 = 0$, and such that the homology of E_r is E_{r+1}

DEFINITION: -

Let F be an object with a differential (i.e,endomorphism) d such that $d^2 = 0$.We assume that F is **filtered** and a sequence is,

$$F = F^0 \supset F^1 \supset F^2 \supset \ldots\ldots. F^n \supset F^{n+1} = \{ 0 \},$$

and that $dF^p \subset F^p$. This data is called a **filtered differential object**.

DEFINITION: -

The **associated graded object,**

$$GrF = \bigoplus_{p \geqq 0} Gr^p F$$

where, $Gr^p F = F^p / F^{p+1}$ and Gr F is a complex with a differential of degree o induced by d itself and the homology $H (Gr^p F)$.

.The pair (c, d) is called the **Bidegree**of f, written by deg (f) =(c,d).

DEFINITION: -

Let$\mathcal{A}$ be an abelian category. E and D objects in $\mathcal{A}$ and

$\alpha : E \longrightarrow E$, $\beta : E \longrightarrow D$and $\gamma : D \longrightarrow E$morphisms.
The data (E, D, α, β, γ) form an **exact couple** if the diagram,

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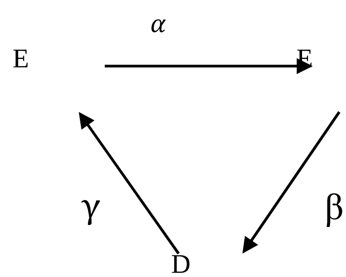


is exact at all three points.

The composite $\gamma \circ \beta$ is zero. Thus, $(\beta\circ \gamma)^2 = 0$. so, there is a differential $d = \beta \circ \gamma : D \longrightarrow D$. The homology of D with respect to this differential d toget,

$$H (E, d) = \ker (d) / \operatorname{im} (d).$$

THEOREM: -

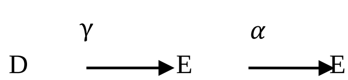
The maps α′ , β′and γ′ defined in a derivedcouple are well defined , and make the derived couple (E′, D′, α′ , β′, γ′) into an exact couple.

PROOF: -

The map $\alpha' : E' \longrightarrow E'$ is clearly well defined , as it is the restriction of a well-definedmap.

If α (x) = α (y)for some x , y ∈ E, then x – y ∈ im(γ) by

exactness of,



If $z, z' \in D$, with $Z' \in im(d)$, wehave $\gamma(z + z') = \gamma(z) + \gamma(z')$. By exactnessat,

$$E \xrightarrow{\beta} D \xrightarrow{\alpha} E$$

$\gamma(z') = 0$, so, γ' is well defined.

Now, we check exactness.

$$\begin{aligned} Ker(\alpha') &= ker(\alpha) \cap im(\alpha) \\ &= im(\gamma) \cap ker(\beta) \\ &= \gamma(\gamma^{-1}(ker\ \beta)) \\ &= \gamma(ker\ d) \\ &= \gamma'(ker\ d / im\ d) \end{aligned}$$

$ker(\alpha') = im\gamma'$.

$$\begin{aligned} ker(\beta') &= \beta^{-1}(im\ d) / ker\ i \\ &= \beta^{-1}(\beta(im\gamma)) \\ &= (im\gamma + ker\ \beta) / im\ \alpha \\ &= (ker\alpha + ker\beta) / ker\alpha \\ &= \alpha(ker\ \beta) \\ &= \alpha(im\ \alpha) \end{aligned}$$

$ker(\beta') = im\ \alpha'$.

$$\begin{aligned} ker(\gamma') &= ker\gamma / imd \\ &= im\ \beta / imd \\ &= \beta E / imd \end{aligned}$$

$ker(\gamma) = im\beta'$.

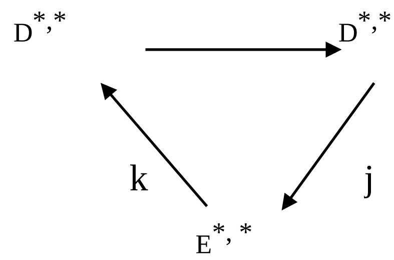
The two starred equalities use the fact that if $i : M \longrightarrow M$ is a self – map of modulus , then

$$lim\, i \cong M / ker\, i.$$

THEOREM: -

Suppose,

i



is an exact triangle of bigraded R – modules, where i, j and k are homomorphisms of bidgrees (-1, 1) , (0, 0), and (1, 0) respectively. These

data give rise to sa spectral sequence $\{ E_r^{*,*}, d_r \}$,where $E_r^{*,*} = (E^{*,*})^{(r-1)}$,

the $(r-1)$-st derived module (i.e., the corresponding module in the derived couple) of $E^{*,*}$ and $d_r = j^{(r-1)} \circ k^{(r-1)}$.

PROOF :

By the definition of derived couple of an exact couple.

We know that, E_{r+1} is the homology of E_r with respect to d_r . It thus suffices to check that the derived differentials d_r have the correct bidegree
(r, 1, -r) which we shown by induction.

Let $E_1 = E^{*,*}$ and $d_1 = j \circ k$. So d_1 has digree (1, 0) (bidegree is additive). Now, suppose by inductionthat$j^{(r-1)}$ has bidegree (r – 2 , 2 – r) and $k^{(r-1)}$ has bidegree (1 , 0).
By definition ,

$$j^{(r)} (i^{(r-1)} (x)) = j^{(r-1)} (x) + d^{(r-1)} E^{(r-1)} ,$$

So the image of $j^{(r)}$ in $(E^{p,q})^{(r)}$ must come from, $i^{(r-1)} (D^{p-r+2, q+r-2})^{(r-1)} = (D^{p-r+1, q+r-1})^{(r)}$

So, $j^{(r)}$ has bidgree($r-1$, $1-r$).
Furthermore as,

$$k^{(r)} (e + d^{(r-1)} E^{(r-1)}) = k^{(r-1)} (e).$$

and $k^{(r-1)}$ has bidegree (1,0). So does$k^{(r)}$.

Thus by induction, we find that $d^{(r)}$ has bidegree(r, 1, -r) and so

$\{ E_r, d_r \}$ is a cohomological spectral sequence.

PROPOSITION: -

In order that the filtration F be weakly convergent it is necessary and sufficient that for each p the intersection of the image of the homomorphisms.

$H (F^{p}A / F^{p+r}A) \longrightarrow H (F^{p+1} A)$ $r \geqq 1$ be zero.

We define,

$$R^{p} = \bigcap_{r} Im \quad (H (F^{p+r}A) \longrightarrow H (F^{p} A)) \quad r > 1$$

$$R^{-\infty} = \bigcap_{p} F^{p} H (A) = \bigcap_{r} Im \quad (H (F^{p} A) \longrightarrow H (A))$$

The homomorphisms H (F^{p+1} A) ⟶ H (F^{p} A) and

H (F^{p}A) → H (A) induce homomorphisms.

$$R^{p+1} \to R^{p} \quad , \quad R^{p} \to R^{-\infty}$$

THEOREM: -

The filtration F is weakly convergent if and onlyifeach $R^{P+1} \longrightarrow R^{p}$ is amonomorphism.

PROOF: -

Consider the diagram,

H (F^{p+1})

H (F^{p}/F^{p+r}) ⟶ H (F^{p+r}) ⟶ H (F^{p})

and let x ∈ R^{p+1}. H (F^{p+r}), and it follows that x is in the image of H (F^{p} / F^{p+r}) if and only if the image of x in R^{p} iszero.

We obtain the relation,

$\bigcap_{r\geq 1} Im$ (H (F^{p}/F^{p+r}) ⟶ H (F^{p+1})) = ker (R^{p+1} ⟶ R^{p})The filtration F of A is convergent if it is weakly convergentand,

$\bigcap_{p} F^{p}$H (A)= 0 (i.e., $R^{-\infty}$ = 0)If we consider thehomomorphism,

u:H(A) ⟶ $\underleftarrow{\lim} H(A)$ H / F^{p} H(A)

definedby

u_p :H(A) H (A) / F^{p} H (A), we find that $R^{-\infty}$ = ker u.

Thus for a convergent filtration, u is a monomorphism. The filtration F of A is strongly convergent, if it is weakly convergent and if u is an isomorphism.

Clearly a strongly convergent filtration is convergent.

Consider the homomorphism,

: $\underleftarrow{\lim} H(A)$ / F^{p} H (A) ⟶ lim H (A / F^{p}A)

induced by,

$$\upsilon_p = H(A) / F^p H(A) \longrightarrow H(A / F^p A)$$

CONCLUSION

In this project concluded that the briefly explained about spectral sequences , and also its used in the concepts are particularly in algebraic topology , algebraic geometry , algebraic number theory , commutative algebra , operator algebras and etc…..

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