A STUDY ON FINITE FREE SOLUTIONS

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ABSTRACT

This article deals with the study of Algebra structures which will tend to finite free resolution of cyclic modules over an area noetherian ring. In recent years, a number of people exploited the unique algebric structure that can be put on a minimal finite free resolution of residue class field of local ring R.

INTRODUCTION

Algebra (from Arabic : al –jabr meaning " reunion of broken parts and bonesetting) is one among the broad parts of mathematics ,together with number theory,geometryandanalysis.initsmostgeneralform,algebraisthestudyof mathematical symbols and the rules for manipulating these symbols ,it is a unifying thread of almost all ofmathematics.

Homologicalalgebraisthebranchofmathematicsthatstudieshomology in a general algebraic setting. Homological algebra began to be studied in its most elementary form within the 1800s as a branch of topology , but it wasn't until the 1940s that it became an independent subject with the study of objects like the ext functor and therefore the tor functor , among others. It is a comparatively young discipline, whose origins are often traced to investigations in combinatorial topology (a precursor to algebraic topology) and abstract algebra (theory of modulesandsyzygies)attheendofthe19thcentury,chieflybyHenripoincare and Davidhilbert. Homological algebra affords the means to extract information contained within the se complexes and present it in the sort of homological invariants of rings ,modules ,topologicalspaces ,andother‘tangible’mathematicalobjects . A powerful tool for doing this provided by spectral sequence

DEFINITION

Let A be a ring.A module E is called **stably free** if there exists a finite free module F such that E$\oplus$F is finite free, and thus isomorphic to $A^{(n)}$ for some positive integer n. In particular, E is projective and finitely generated.

We say that a module M has a **finite free resolution** if there exist a resolution ,

$$0 \longrightarrow E_n \longrightarrow \cdots\cdots E_0 \longrightarrow M \longrightarrow 0.$$

Such that each E_i is finite free.

THEOREM

Let M be a projective module. Then M is stably free if and only if M admits a finite free resolution .

PROOF:

If M is stably free then it is trivial that M has a finite free resolution . conversely assume the existence of the resolution with the above notation. We prove that M is stably free by induction on n.

The assertion is obvious if n =0. Assume n $\geqq$ 1. Insert the kernals and cokernals at each step,in the manner of dimension shifting ,say

$$M_1 = \mathrm{Ker}(\ E_0 \longrightarrow P)$$

Giving rise to the exact sequence

$$0 \longrightarrow M_1 \longrightarrow E_0 \longrightarrow M \longrightarrow 0.$$

Since , M is projective this sequence splits , and $E_0 \approx M \oplus M_1$. But M_1 has a finite free resolution of length smaller than the resolution of M, so there exists a finite free module F such that $M_1 \oplus F$ is free. Since $E_0 \oplus F$ is also free.

A resolution ,

$$0 \longrightarrow E_n \longrightarrow E_0 \longrightarrow M \longrightarrow 0$$

is called stably free if all the modules E_i(i= 0,……n) are stably free.

PROPOSITION

Let M be an A- module. Then M has a finite free resolution of length $n \geqq 1$ if and only if M has a stably free resolution of length n.

PROOF:

One direction is trivial, so we suppose given a stably free resolution with the above notation.

Let $0 \leqq i < n$ be some integer, and let F_i, F_{i+1} be finite free such that $E_i \oplus F_i$ and $E_{i+1} \oplus F_{i+1}$ are free.

Let $F = F_i \oplus F_{i+1}$. Then we can form an exact sequence,

$$0 \to E_i \to \cdots\cdots \to E_{i+1} \oplus F \to E_i \oplus F \to \cdots\cdots \to E_0 \to M \to 0$$

In the obvious manner. In this way, we have changed two consecutive modules in the resolution to make them free. Proceeding by induction, we can then make E_0 , E_1 free, then E_1 , E_2 free and so on .

THEOREM

Let M be a module which admits a free resolution of length n.

$$0 \longrightarrow E_n \longrightarrow \cdots\cdots \longrightarrow E_0 \longrightarrow M \longrightarrow 0$$

PROOF

Let $F_m \longrightarrow \cdots\cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0$

be an exact sequence with F_i stably free for i = 0,… .... ,m.

(i) If $m < n-1$ then there exists a stably free F_{m+1} such that exact sequence can be continued exactly to

$$F_{m+1} \longrightarrow \dots\dots\dots \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

(ii) If $m = n-1$,let $F_n = \ker (F_{n-1} \longrightarrow F_{n-2})$. Then F_n is stably free and thus $0 \longrightarrow F_n \longrightarrow$

$$F_{n-1} \longrightarrow \dots\dots\dots\dots\dots\dots \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

is a stably free resolution.

DEFINITION

The minimal length of a stably free resolution of a module is called its **stably free dimension**.

COROLLARY

If $0 \longrightarrow M_1 \longrightarrow E \longrightarrow M \longrightarrow 0$ is exact , M has stably free dimension $\leqq n$, and E is stably free ,then M_1 has stably free dimension $\leqq n-1$.

DEFINITION

Let A be a ring and M a module .A sequence of elements $x_1,\dots ,x_r$ in a A is called M-regular if $M / (x_1,\dots,x_r) M \neq 0$, if x_1 is not divisor of zero in M , and for $i \geqq 2$, x_1 is not divisor of 0 in

$$M / (x_1,\dots ,x_{i-1}) M$$

It is called **regular** when M = A.

DEFINITION

Let A be a commutative ring and $x \in A$. we define the complex K(x) to have $K_0(x) = A$, $K_1(x) = Ae_1$, where e_1 is a symbol , Ae_1 is the free module of rank 1 with basis $\{e_1\}$, and the boundary map is defined by $de_1 = x$, so the complex can be represented by the sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & Ae_1 & \xrightarrow{d} & A & \longrightarrow & 0 \\ & & \| & & \| & & \\ 0 & \longrightarrow & K_1(x) & \longrightarrow & K_0(x) & \longrightarrow & 0 \end{array}$$

More generally, for elements $x_1,\ldots\ldots,x_r \in A$ we define the **Koszul complex** $K(x) = K(x_1,\ldots\ldots,x_r)$ as follows .we put;

$K_0(x) = A$;

$K_1(x)$ = free module E with basis $\{ e_1,\ldots..,e_r)$;

$K_p(x)$ = free module $\wedge^p E$ witth basis $\{ e_{i_1} \wedge \ldots . \wedge e_{i_p} \}$, $i_1 < \cdots\cdots < i_p$;

$K_r(x)$ = free module $\wedge^r E$ of rank 1 with basis $e_1 \wedge \cdots \wedge e_r$.

And we define the boundary maps by $de_i = x_i$ and general $d : K_p(x)$ $\longrightarrow K_{p-1}(x)$ by,

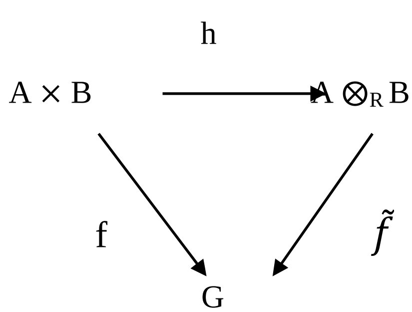
$$d (e_{i_1} \wedge \ldots . \wedge e_{i_p}) = \sum_{j=1}^{p} (-1)^{j-1} x_{i_j} \, e_{i_1} \wedge \cdots \wedge \hat{e}_{i_j} \wedge \cdots \wedge e_{i_p} \}$$

DEFINITION

Given a ring R and modules A_R and ${}_R B$, then their **tensor product** is an abelian group $A \otimes_R B$ and an R- biadditive function

$$h : A \times B \longrightarrow A \otimes_R B \text{ such that },$$

for every abelian group G and every R-biadd $f : A \times B \longrightarrow G$, there exists a unique $\mathbb{Z}$ - homomorphism $\tilde{f} : A \otimes_R B \longrightarrow G$ making the following diagram commute:



THEOREM

Let $D = \det(c_{ij})$ be the determinant. then for $p = r$ we get that $f_r : K_r(y) \longrightarrow K_r(x)$ is multiplication by D,

the homomorphisms f_p define a morphism of Koszul complexes :

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & K_r(y) & \longrightarrow \cdots \longrightarrow & K_p(y) & \longrightarrow \cdots \longrightarrow & K_1(y) & \longrightarrow & A & \longrightarrow & A/I' & \longrightarrow & 0 \\
 & & \downarrow {\scriptstyle fr = D} & & \downarrow {\scriptstyle f_p} & & \downarrow {\scriptstyle f_1} & & \downarrow {\scriptstyle id} & & \downarrow {\scriptstyle can} & & \\
0 & \longrightarrow & K_r(x) & \longrightarrow \cdots \longrightarrow & K_p(x) & \longrightarrow \cdots \longrightarrow & K_1(x) & \longrightarrow & A & \longrightarrow & A/I & \longrightarrow & 0
\end{array}$$

and define an isomorphism if D is a unit in A , for instance if (y) is a permutation of (x) .

PROOF

A complex

$$0 \longrightarrow K_r(x) \longrightarrow \cdots\longrightarrow K_p(x) \longrightarrow \cdots \longrightarrow K_1(x) \longrightarrow A \longrightarrow 0 \text{ is}$$

independent of the ideal $I = (x_1 ,\ldots, x_r)$ generated by (x). Let

$$I = (x_1 ,\ldots, x_r) \supset I' = (y_1 ,\ldots, y_r)$$

be two ideals of A . We have a natural ring homomorphism can : $A/I' \longrightarrow A/I$.

Let $\{ e'_1,\ldots, e'_r\}$ be a basis for $K_1(y)$, and let

$$y_i = \sum c_{ij}\, x_{ij} \qquad \text{with} \qquad c_{ij} \in A.$$

we define $f_1 : K_1(y) \longrightarrow K_1(x)$ by

$$f_1 e'_i = \sum c_{ij}\, e_j$$

and $f_p = f_1 \wedge \cdots \wedge f_1$, product taken p times. By definition,

$$f (e'_{i1} \wedge \cdots \wedge e'_{ip}) = \Big(\sum_{j=1}^{r} c_{i_1 j}\, e_j \Big) \wedge\cdots\wedge \Big(\sum_{j=1}^{r} c_{i_p j}\, e_j \Big)$$

then

$$f\,d\,(e'_{i_1}\wedge\cdots\wedge e'_{i_p}) = f\,\Big(\sum_k (-1)^{k-1}\, y_{i_k}\, e'_{i_1}\wedge\cdots\wedge \hat{e}'_{i_k}\wedge\cdots\wedge e'_{i_p}\Big)$$

$$= \sum_k (-1)^{k-1}\, y_{i_k}\Big(\sum_{j=1}^{r} c_{i_1 j}\, e_j\Big)\wedge\cdots\wedge \hat{\sum_k}\wedge\cdots\wedge\Big(\sum_{j=1}^{r} c_{i_p j}\, e_j\Big)$$

$$= \sum (-1)^{k-1}\Big(\sum_{j=1}^{r} c_{i_1 j}\, e_j\Big)\wedge\cdots\wedge\underbrace{\Big(\sum_{j=1}^{r} c_{i_k j}\, x_j\, e_j\Big)}_{\text{omitted}}\wedge\cdots\wedge\Big(\sum_{j=1}^{r} c_{i_p j}\, e_j\Big)$$

$$= d\,f\,(e'_{i_1}\wedge\cdots\wedge e'_{i_p})$$

Using $y_{i_k} = \sum c_{i_k j}\, x_j$. This concludes the proof that the f_p define a homomorphism of complexes.

In particular , if (x) and (y) generate the same ideal , and the determinant D is a unit (i.e. the linear transformation going from (x) to (y) is invertible over the ring) , then the two Koszul complexes are isomorphic.

THEOREM

There is a natural isomorphism

$$K(x_1,\ldots,x_r) \approx K(x_1)\otimes\cdots\otimes K(x_r) .$$

PROOF:

Let $I = (x_1,\ldots,x_r)$ be the ideal generated by $x_1,\ldots,x_r$. then directly from the definitions of the 0-th homology of the Koszul complex is A / IA .

More generally , let M be an A-module.Define the Koszul complex of M by ,

$$K(x ; M) = K(x_1,\ldots,x_r ; M) = K(x_1,\ldots,x_r)\otimes_A M$$

Then this complex looks like

$$0 \longrightarrow K_r(x)\otimes M \longrightarrow \cdots\longrightarrow K_2(x)\otimes_A M \longrightarrow M \longrightarrow 0 .$$

We sometimes abbreviate $H_p(x ; M)$ for $H_p K(x ; M)$. The first and last homology groups are then obtained directly from the definition of boundary. We get,

$$H_0\,(K(x ; M)) \approx M / IM$$

$H_r\,(K(x) ; M) = \{ v \in M$ such that $x_i v = 0$ for all $i = 1,\ldots r \}$.A tensor product of any complex with K(x) , when x consists of a single element. Let y $\in$ M and let C be an arbitrary complex of A-modules. We have an exact sequence of complexes

$$0 \longrightarrow C \longrightarrow C \otimes K(y) \longrightarrow (C \otimes K(y)) / C \longrightarrow 0 \qquad \longrightarrow (1)$$

Made explicit as follows,

$$\begin{array}{ccccccccc}
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_{n+1} & \longrightarrow & (C_{n+1} \otimes A) \oplus (C_n \otimes K_1(y)) & \longrightarrow & C_n \otimes K_1(y) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_n & \longrightarrow & (C_n \otimes A) \oplus (C_{n-1} \otimes K_1(y)) & \longrightarrow & C_{n-1} \otimes K_1(y) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_{n-1} & \longrightarrow & (C_{n-1} \otimes A) \oplus (C_{n-2} \otimes K_1(y)) & \longrightarrow & C_{n-2} \otimes K_1(y) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & &
\end{array}$$

We note that $C \otimes K_1(y)$ is just C with a dimension shift by one unit , in other words,

$$(C \otimes K_1(y))_{n+1} = C_n \otimes K_1(y) \qquad \longrightarrow (2)$$

In particular,

$$H_{n+1}(C \otimes K(y) / C) \approx H_n(C) \qquad \longrightarrow (3)$$

Associated with an exact sequence of complexes, we have the homology sequence , which in this case yields the long exact sequence ,

$$\longrightarrow H_{n+1}(C) \longrightarrow H_{n+1}(C \otimes K_1(y)) \longrightarrow H_{n+1}(C \otimes K(y)/C) \xrightarrow{\partial} H_n(C)$$

$$H_{n+1}(C \otimes K(y)/C) \approx H_n(C)$$

Which we write stacked up according to the index :

$$\longrightarrow H_{p+1}(C) \longrightarrow H_{p+1}(C) \longrightarrow H_{p+1}(C \otimes K(y)) \longrightarrow$$
$$\longrightarrow H_p(C) \longrightarrow H_p(C) \longrightarrow H_p(C \otimes K(y)) \longrightarrow \qquad (4)$$

Ending in lowest dimension with

$$\longrightarrow H_1(C) \longrightarrow H_1(C \otimes K(y)) \longrightarrow H_0(C) \longrightarrow H_0(C) \longrightarrow \qquad (5)$$

Furthermore , a direct application of the definition of the boundary map and the tensor product of complexes yields :

The boundary map on $H_p(C)$ ($p \geqq 0$) is induced by multiplication by ($-1)^p$ y :

$$\partial = (-1)^p m(y) : H_p(C) \longrightarrow H_p(C) \longrightarrow \qquad (6)$$

sssIndeed , write

$$(C \otimes K(y))_p = (C_p \otimes A) \oplus (C_{p-1} \otimes K_1(y)) \approx C_p \oplus C_{p-1}.$$

Let $(v,w) \in C_p \oplus C_{p-1}$ with $v \in C_p$ and $w \in C_{p-1}$. then directly from the definitions,

$$d(v,w) = (dv + (-1)^{p-1} yw , dw) \longrightarrow \qquad (7)$$

To see (6) , one merely follows up the definitions of the boundary , taking an element $w \in C_p \approx (C_p \otimes K_1(y))$, lifting back to (0,w) , applying d , and lifting back to C_p.

If we start with a cycle ,i.e. $dw = 0$,

then the map is well defined on the homology class , with values in the homology.

DEFINITION

EXT FUNCTOR:

Let R be a ring and let $\mathcal{a}$ = Mod (R) be the category of R-modules. Fix a module A. The functor $M \mapsto Hom(A, M)$ is left exact, i.e. given an exact sequence $0 \longrightarrow M' \longrightarrow M \longrightarrow M''$, the sequence

$$0 \longrightarrow Hom(A, M') \longrightarrow Hom(A, M) \longrightarrow Hom(A, M'')$$

is exact. Its right derived functors are denoted by $Ext^n(A, M)$ for M variable.

DEFINITION

TOR FUNCTOR :

Let R be commutative. The functor $M \mapsto A \otimes M$ is right exact , in other words , the sequence

$$A \otimes M' \longrightarrow A \otimes M \longrightarrow A \otimes M'' \longrightarrow 0$$

is exact. Its left derived functors are denoted by $Tor_n(A, M)$ for M variable.

THEOREM

Let $x_1,\ldots\ldots,x_r$ be an M-regular sequence in A . let $I = (x)$. then $Ext^i(A/I, M) = 0$ for $i < r$.

PROOF:

we assume that the exact homology sequence. Assume by induction that $Ext^i(A/I, M) = 0$ for $i < r-1$.then we have the exact sequence,

$$0 = Ext^{i-1}(A/I, M/x_1M) \longrightarrow Ext^i(A/I, M) \longrightarrow Ext^i_{X1}(A/I, M)$$

for i < r. But $x_1 \in I$ so multiplication by x_1 induces 0 on the homology groups, which gives $Ext^i(A/I, M) = 0$ as desired.

Let $L_n \longrightarrow N \longrightarrow 0$ be a free resolution of a module N.by definition,

$Tor_i^A(N, M)$ = i-th homology of the complex L⊗M. This is independent of the choice of L_N up to a unique isomorphism.

COROLLARY

If x is a regular sequence in R ,then K(x) is a free resolution of R/I, Z = ($x_1, \ldots, x_n$) R. that is , the following sequence is exact :

$$0 \longrightarrow \Lambda^n(R^n) \longrightarrow \cdots\cdots \longrightarrow \Lambda^2(R^n) \longrightarrow R^n \overset{x}{\longrightarrow} R \longrightarrow R/I \longrightarrow 0$$

In this case we have,

$Tor_p^R(R/I, A) = H_p(x, A)$:

$Ext_R^p(R/I, A) = H_p(x, A)$.

CONCLUSION

In this project concluded that the briefly explained about HOMOLOGICAL ALGEBRA , and also its utilized in the concepts are particularly in algebraic topology , algebraic geometry , irrational number theory , commutative algebra , operator algebras and etc…..

BIBLIOGRAPHY

1. **Serge Lang ,** " Algebra " revised third edition , springer , 2002.
2. **J.Rotman ,** " Introduction to homological algebra ", Academic press, 1979.
3. **D.Northcott ,** " Finite free resolutions ", Cambridge University Press , 1976.
4. **H.Cartan and S.Eilenberg** , " Homological algebra ", Princeton university press , 1957.
5. **Charles A. weibel** , " An introduction to homological algebra", Cambridge university press, 1994.
6. **Anthony W. Knapp** , "Advanced algebra ",digital second edition, 2016.
7. **S.Arumugam and A.Thangapandi Isaac** , "Modern algebra ", Edition 2000.