A STUDY ON COHOMOLOGY RINGS

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ABSTRACT:

Cohomology is a general term for a sequence of abelian groups defined from a cochain complex. That is, cohomology is defined as the abstract study of cochains, cocycles and coboundaries. Cohomology can be viewed as a method of assigning algebraic invariants to a topological space that has a more refined algebraic structure than the homology.

KEYWORDS: Cohomology ring, Subring

Definition:

In mathematic specifically algebraic topology the cohomology ring of a topological space X is ring formed the cohomology groups of X together with the cup product serving as the ring multiplication.

Here cohomology is usually understood as singular cohomology but the ring structure is also present in other theories such as de Rhamcohomology. It is also functorial for a continuous mapping of spaces one obtains a ring homomorphism on cohomology rings which is contra variant.

Specifically gives sequences of cohomology groups $H^K(X; R)$ on X with coefficient in a commutative ring R one can define the cup product which takes the form.

$$H^K(X; R) \times H^l(X; R) \longrightarrow H^{K+l}(X; R)$$

The cup product gives a multiplication on the direct sum of the cohomology groups.

$$\dot{H}(X; R) = \bigoplus_{K \in N} H^K(X; R)$$

Definition:

A ring R is a graded ring if there are additive subgroups R^n.n≥0 such that

i) $R = \sum_{n\geq 0} R^n$ (direct sum of additive groups)

ii) $R^n R^m \subset R^{n+m}$ for all n, m ≥ 0 that is if $x \in R^n$ and $y \in R^m$, then $xy \in R^{n+m}$

Definition:

An element x in a graded ring $R = \Sigma R^n$ has degree n if $x \in R^n$ such element are called homogeneous. A (tow sided) ideal I (or) a subring S is called homogeneous if it is generated by homogeneous elements. $I = \Sigma (I \cap R^n)$

Lemma:

If I is a homogeneous ideal in a graded ring $R = \Sigma R^n$ is graded ring indeed

$$R / I = \Sigma (R^n + I) / (I)$$

Proof:

Given that if I is a homogeneous ideal
To prove that $R / I = \Sigma (R^n + I) / (I)$

$$R = \Sigma R^n$$

Since I is homogeneous $I = \Sigma (I \cap R^n)$
As abelian groups $R / I = \Sigma R^n / \Sigma (I \cap R^n)$

$$\cong \Sigma (R^n / I \cap R^n)$$

$$\cong \Sigma (R^n + I / I)$$

Also
$(R^n + I) / (I) . (R^m + I) / (I) \subset (R^n R^m + I) / (I)$ [∵ because I is an ideal]
$(R^n R^m + I) / I \subset (R^{n+m} + I) / I$
$\therefore R / I = \Sigma (R^n + I) / (I)$

Definition:

Let x be a space and let R be a commutative ring. If $\emptyset \in S^n (X, R)$ and $\emptyset \in S^m (X, R)$, defined their cup product $\emptyset \cup \theta \in S^{n+m}(X, R)$ by

$(\sigma \emptyset \cup \theta) = (\sigma\lambda_n , \phi) (\sigma\mu_m, \theta)$ for every (n+m) simplex σ in X, where right side is the product of two element in the Ring R of course, cup product defines a function

$S^*(X, R) \times S^*(X, R) \longrightarrow S^*(X, R)$ defining

$(\Sigma\phi_i) \cup (\Sigma\phi_j) = \sum_{i,j} \cup \phi_i$

Where $\emptyset_i \in S^i(X, R)$ is a graded ring under cup product

Lemma:

If X is a space and R is a commutative ring then $S^*(X, R) = \Sigma S^n(X, R)$ is graded ring under cup product

Proof:

To prove left distributivity

To show that $\emptyset \cup (\theta + \psi) = (\emptyset \cup \theta) + (\emptyset \cup \psi)$

When $\emptyset \in S^n(X, R)$ and $\theta, \psi \in S^m(X, R)$

But if σ is an (n + m) – simplex

$(\sigma, \emptyset \cup (\theta + \psi)) = (\sigma\lambda_n, \emptyset)(\sigma\mu_m, \theta + \psi)$

$= (\sigma\lambda_n, \emptyset)[(\sigma\mu_m, \theta) + (\sigma\mu_m, \psi)$

$= (\sigma, \emptyset \cup \theta) + (\sigma, \emptyset \cup \psi)$

To prove that right distributivity

To show that $(\theta + \psi) \cup \emptyset = (\theta \cup \emptyset) + (\psi \cup \emptyset)$

$(\sigma, (\theta + \psi) \cup \emptyset) = (\sigma\lambda_n, (\theta + \psi)(\sigma\mu_m, \theta)$

$= [(\sigma\lambda_n, \theta) + (\sigma\lambda_n, \psi)](\sigma\mu_m, \theta)$

$= (\sigma, \theta \cup \psi) + (\sigma, \psi \cup \theta)$

To prove associativity

Let $\emptyset \in S^n(X, R)$, $\theta \in S^m(X, R)$ and $\psi \in S^K(X, R)$

If σ is an (n + m + K) – simplex.

Then $(\sigma, \emptyset \cup (\theta \cup \psi)) = (\sigma\lambda_n, \emptyset)(\sigma\mu_{m+k}\lambda_m, \theta)(\sigma\mu_{m+k}\mu_k, \psi)$

And $(\sigma, (\emptyset \cup \theta) \cup \psi) = (\sigma\lambda_{n+m}\lambda_n, \emptyset)(\sigma\lambda_{n+m}\mu_m, \theta)(\sigma\mu_k, \psi)$

These two products are equal

Define $e \in S^\circ (X, R)$ by $(x, e) = 1$ for all $x \in X$

It is easy to see that e is an identity in $S^* (X, R)$

Hence $S^* (X, R)$ is a ring

$S^* (X, R)$ is graded ring

Lemma:

If $f: X \to X'$ is continuous map

Then $f^\# (\emptyset \cup \theta) = f^\# (\emptyset) \cup f^\# (\theta)$

Moreover, if $e \in S^\circ (X, R)$ is the unit ($(x, e) = 1$ for all $x \in X$) and if $e' \in S^\circ (X', R)$ is defined by $(x', e') = 1$ for all $x' \in X$, then $f^\# (e') = e$

Proof:

Assume that $\emptyset \in S^p (X', R)$ and $\theta \in S^q (X', R)$

Now if σ is a (p + q) _ simplex in X.

$(\sigma, f^\# (\emptyset \cup \theta)) = (f\,\sigma, (\emptyset \cup \theta)$

$= (f\,\sigma\lambda_p, \emptyset)\,(f\,\sigma\,\mu_q, \theta)$

$= (\sigma\lambda_p, f^\# \emptyset)\,(\sigma\,\mu_q, f^\# \theta)$

$= (\sigma, f^\# \emptyset \cup f^\# \theta)$

$f^\# (\emptyset \cup \theta) = f^\# (\emptyset) \cup f^\# (\theta)$

if $x \in X$, then $(x, f^\# (e')) = (f(x), e') = 1$

Lemma:

If $\emptyset \in S^p (X, R)$ and $\theta \in S^q (X, R)$, then

$\delta(\emptyset \cup \theta) = \delta\emptyset \cup \theta + (-1)^p \emptyset \cup \delta\theta$

Proof:

Note that both sides have degree $d = p + q + 1$

If σ is a d – simplex

Then $(\sigma, \delta\emptyset \cup \theta + (-1)^p \emptyset \cup \delta\,\theta)$

$= (\sigma\lambda_{P+1}, \delta\emptyset)\,(\sigma\,\mu_q, \theta) + (-1)^p\,(\sigma\lambda_p, \emptyset)\,(\sigma\mu_{q+1}, \delta\theta)$

$= (\partial\ (\sigma\lambda_{P+1}), \emptyset)\ (\sigma\mu_q, \theta) + (-1)^p\ (\sigma\lambda_p, \emptyset)\ (\partial(\sigma\mu_{q+1}, \theta)$

$= \sum_{i=0}^{p+1}(-1)^i\ (\sigma\ \lambda_{P+1} \in_i, \emptyset)\ (\sigma\mu_q, \theta) + \sum_{j=0}^{q+1}(-1)^{j+p}\ (\sigma\lambda_p, \emptyset)\ (\sigma\mu_{q+1}\in_j, \theta)$

$\sigma\lambda_{P+1}\in_{P+1} = \sigma\lambda_{P+1}\lambda_P = \sigma\lambda_P$ and $\sigma\mu_{q+1}\in_0 = \sigma\ \mu_{q+1}\mu_q = \sigma\ \mu_q$

If follows that terms P+1 of the first sum cancels terms 0 of the second sum and so the two sums equal

$\sum_{i=0}^{p}(-1)^i\ (\sigma\ \lambda_{P+1} \in_i, \emptyset)\ (\sigma\mu_q, \theta) + \sum_{j=1}^{q+1}(-1)^{j+p}\ (\sigma\lambda_p, \emptyset)\ (\sigma\mu_{q+1}\in_j, \theta)$

On the other hand

$(\sigma, \delta(\emptyset \cup \theta) = (\ \partial\ \sigma, \emptyset \cup \theta)$

$= \sum_{i=0}^{d}(-1)^i\ (\sigma \in_i, \emptyset \cup \theta)$

$= \sum_{i=0}^{d}(-1)^i\ (\sigma \in_i\lambda_p, \emptyset)\ (\sigma \in_i\mu_q, \theta)$

$= \sum_{i=0}^{D}(-1)^i(\sigma \in_i\lambda_p, \emptyset)\ (\sigma \in_i, \theta) +$
$\sum_{i=P+1}^{d}(-1)^i(\sigma \in_i\lambda_p, \emptyset)\ (\sigma \in_i \mu_q, \theta)$

Since d – q = p + 1

$\sum_{i=0}^{D}\ (-1)^i\ (\sigma\lambda_{p+1} \in_i, \emptyset)\ (\sigma\mu_q, \theta) + \sum_{i=P+1}^{d}(-1)^i\ (\sigma\lambda_p, \emptyset)\ (\sigma\mu_{q+1} \in_{i-p}, \theta)$

But the index of summation in the second sum can be changed to j = i – p giving $\sum_{j=1}^{q+1}\ (-1)^{j+p}(\ \sigma\lambda_p, \emptyset)\ (\sigma\mu_{q+1}\in_j, \theta)$ as desired.

Theorem:

For any commutative ring R, $H^*(\ ;\ R) = \sum_{P\geq 0} H^i(\ ;\ R)$ is a contra variant functorhTop Graded → Rings.

Proof:

Let $Z^*(X, R) = \Sigma\ Z^P(X, R)$ and

$B^*(X, R) = \Sigma\ B^P(X, R)$

If $\phi\epsilon Z^P$ and $\theta\epsilon Z^q$, then $\delta\phi = 0 = \delta\theta$

And $\delta(\phi \cup \theta) = \delta\phi \cup \theta + (-1)^P \emptyset \cup \delta\theta = 0$

Hence $\phi \cup \theta$ is a cocycles.

If follows that Z^* is a subring of $S^*(X, R)$

If $\phi \in Z^P$ and $\theta \in B^q$

Then $\delta\phi = 0$ and $\theta = \delta\Psi$ for some $\Psi \in S^{q-1}$ (X, R)

$$\text{Hence } \phi \cup \theta = \phi \cup \delta\,\Psi = \pm\,(\delta(\emptyset \cup \psi) - \delta\emptyset \cup \psi) = \pm\delta(\emptyset \cup \psi)$$

So that $\phi \cup \theta$ is a Coboundary

Similarly, $\theta \cup \phi$ is coboundary

If follows that B^* is two sided homogeneous ideal in Z^*

$H^*(X; R) = Z^* / B^*$ is a graded ring

That a continuous map f: X → Y Yields a ring homomorphism

$F^*: H^*(Y; R) \rightarrow H^*(X; R)$

namely $f^* cls\emptyset = cls\, f^*\emptyset$

Definition:

The multiplication $H^*(X; R) \otimes H^*(X; R) \rightarrow H^*(X; R)$ is also called cup product[1] and one defines

$$cls\emptyset \cup cls\ \theta = cls\ (\emptyset \cup \theta)$$

Theorem:

Cup product is the composite $d^{\#}\zeta^{\#}\pi$

$$S^*(X, R) \otimes S^*(X, R) \rightarrow Hom\ (S_*(X) \otimes S_*(X), R) \rightarrow S^*(X \times X, R) \rightarrow S^*(X, R)$$

Proof:

Let ϕ, $d^{\#}\zeta^{\#}\phi \otimes \theta = (\zeta d_{\#}\sigma, \phi \otimes \theta)$

$$= (\zeta(d\,\sigma), \phi \otimes \theta)$$

Now the Alexander – whitney formula gives

$$\zeta(d\,\sigma) = \sum_{i=0}^{n+m} (d\sigma)'\lambda_i \otimes (\sigma\sigma)''\mu_{n+m-i}$$

Recall that $(d\sigma)' = \pi' d\sigma$ and $(d\sigma)'' = \pi' d\sigma$

Where π' and π'' are projection of $X \times X$ onto the first and second factors.

Since $d: X \rightarrow X \times X$ is the diagonal however both $\pi' d$ and $\pi'' d$ equal the identity on X.

$$\text{Hence } \zeta(d\,\sigma) = \sum_{i=0}^{n+m} \sigma\lambda_i \otimes \sigma\mu_{n+m-i}$$

But $\phi \otimes \theta$ vanishes off $S_n(X) \otimes S_m(X)$.

So that, $(\zeta(d\,\sigma), \phi \otimes \theta) = (\sigma\lambda_n \otimes \sigma\mu_m, \phi \otimes \theta)$

$= (\sigma\lambda_n, \phi)\,(\sigma\,\mu_m, \theta)$

$= (\sigma, \phi \cup \theta)$.

Theorem: [K$\ddot{u}$nneth formula for cohomology]

If x and y are space of finite type then there is a split short exact sequence

$$0 \rightarrow \sum_{i+j=n} H^i A^j(Y) \rightarrow H^n(X \times Y) \rightarrow \sum_{p+q=n+1} \text{Tor}\,(H^p(X), H^q(Y)) \rightarrow 0$$

Where α' : $\text{cls}\emptyset_i \otimes \text{cls}\theta_j \mapsto \text{cls}\ \zeta^{\#}(\emptyset_i \otimes \theta)$

[ζ is an Eilen berg–ziber chaing equivalence $S_*(X \times Y) \rightarrow S_*(X) \otimes S_*(Y)$]

Proof:

Let X and Y have finite type lemma "if X is a space of finite type then there exists a free chain complex C_* of finite type such that C_* is chain equivalent to $S_*(X)$ chain complexes.

C* And E* of finite type chain equivalent $S_*(X)$ and $S_*(Y)$ respectively

The commutative diagram

$$\begin{array}{ccc} H_i(X) \otimes H_j(Y) & \xrightarrow{\alpha'} & H^{i+j}(X \times Y) \\ \downarrow & & \downarrow \\ H^i(C_*) \otimes H^j(E_*) & \xrightarrow{\alpha'} & H^{i+j}(\text{Hom}\,(C_* \otimes E_*, Z)) \end{array}$$

With vertical map is isomorphism

Apply universal coefficient theorem for chohomology.

Both Hom (C*, Z) and Hom (E*, Z) are free chain complexes because C* and E* finite type.

Hence the proof

CONCLUSION:

To combine both the topology and the algebra, it has a variety of possibilities. Universal coefficient theorem, kunneth formula, de Rham cohomology, commutative diagrams are present with the example.

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