A STUDY ON COHOMOLOGY GROUPS

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ABSTRACT:

In this chapter we shall apply the theory of derived functors to the important special case where the cochain complex, sequence of abelian, n – cochain, cyclic, Rham cohomology. This will lead cohomology group $H^n(X, G)$. In developing the theory we shall attempt to deduce as much as possible form general properties of derived functors. Thus or example we shall give a proof of the fact that $H^n(X, G)$.

KEY WORDS:

Singular complex, cohomology class, Abelian group, n boundaries.

INTRODUCTION:

Homology groups Hn (X) are the result of a two-stage process: First one forms a chain complex …. $\longrightarrow C_n \xrightarrow{\partial} C_{n-1} \longrightarrow$ …… of singular, simplicial, or cellular chains, then one takes the homology groups of this chain complex, Ker $\partial / Im\, \partial$. To obtain the cohomology groups H_n (X, G) we interpolate an intermediate step, replacing the chain groups C_n by the dual groups Hom (C_n, G) and the boundary maps ∂ by their dual maps δ, before forming the cohomology groups Ker $\partial / Im\, \partial$. The plan for this section is first to sort out the algebra of this dualization process and show that the cohomology groups are determined algebraically by the homology groups, though in a somewhat subtle way. Then after this algebraic excursion we will define the cohomology groups of spaces and show that these satisfy basic properties very much like those for homology. The payoff for all this formal work will begin to be apparent in subsequent sections.

INCEPTIONS

Definition:

A topology on a set X is a collection τ of subsets of X having the following properties is called a topological space.

i) $\emptyset$ and X are in τ
ii) The union of the element of any sub collection of τ is in τ

iii) The intersection of the elements of any finite sub collection of τ in in τ.

Definition:

A group G is said to be abelian if $ab = ba$ for all a, b ∈ G. A group which is not abelian is called non - abelian group.

Definition:

Let B be a subset of an abelian group F. then F is free abelian with basis B if the cyclic subgroup $< b >$ is infinite cyclic for each $b \in B$ and $F = \sum_{b \in B} < b >$ (direct sum).

A free abelian group is thus a direct sum of copies of Z. A typical element X∈ F has a unique expression.

$X = \Sigma m_b b$

Where $m_b \in Z$ and almost all m_b are zero.

Definition:

A homomorphism f: G $\longrightarrow$ G' is a map such that f (x, y) = f (x).f (y) for all x, y.

It automatically satisfies the equation f (e) = e' and $f(x^{-1}) = f(x)^{-1}$

Where e and e' are the identities of G and G'.

Definition:

For each $n \geq 0$ the n^{th} (singular) homology groups of a space X is

$$H_n(X) = \frac{Z_n(X)}{B_n(X)} = \frac{Ker\partial n}{im\ \partial_{n+1}}$$

Definition:

The cohomology of x written by $H^*(x) = \{H^n(X)\}_{n=-\infty}^{n=+\infty}$ to be the sequence of modules in c given by $H_n(X) = \frac{Ker\ \partial n}{im\ \partial_{n-1}}$ the $Ker\ \partial n$ is called a n-cocycles and the im $\partial_{n-1}$is called n – coboundaries.

Definition:

A chain complex (A•, d•) is a sequence of abelian group or modules …..A_0, A_1, A_2, A_3,…….. connected by a homomorphism [called boundary operatos or differentials] d_n: $A_n \longrightarrow A_{n-1}$ such that the composition of any two consecutive maps is the zero. Explicitly the differential satisfy $d_n\, d_{n+1} = 0$ (or) with indices suppressed $d^2 = 0$ the complex may be written out as follows.

$\ldots\ldots \quad A_0 \quad A_1 \qquad A_2 \quad A_3 \qquad \ldots\ldots$

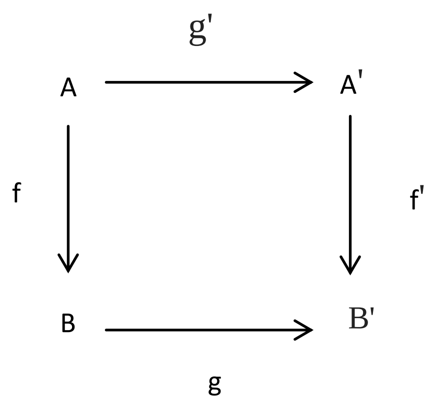
Definition:

The cochain complex $(A^{\bullet}, d^{\bullet})$ is the dual notation to a chain complex. It consists of a sequence of abelian groups or modules $\ldots\ldots A^0, A^1, A^2, A^3\ldots\ldots$ connected by homomorphism $d^n: A^n \longrightarrow A^{n+1}$ satisfying $d^{n+1} \circ d^n = 0$. The cochain complex may be written out in similar fashion to the chain complex.

$$\ldots\ldots \xrightarrow{d^{-1}} A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} A^2 \xrightarrow{d^2} \ldots\ldots$$

Definition:

A commutative diagram in $\mathcal{C}$ is a diagram in which each pair of vertices and every two paths (Composites) between them are equal as morphisms. Also it satisfies the commutative property. That is $g \circ f = f' \circ g'$.



Definition:

An element of H^n (X, G) is a cosect $\zeta + B^n$ (X, G) where ζ is an n – cocycles it is called a cohomology class and it is denoted by cls ζ.

Definition:

A connected open set x in R^n thus determines a sequence of homomorphism

$$0 \longrightarrow \Omega^o(X) \xrightarrow{d^o} \Omega^1(X) \xrightarrow{d^1} \ldots\ldots \; \Omega^n \xrightarrow{d^{n-1}} \qquad 0 \longrightarrow$$

Moreover, there is a straight forward computation showing that dd = 0. In other words this sequence is a complex its homology groups are called the de Rham cohomology of X.

Definition:

If K is an oriented simplical complex and G is an abelian group then the simplical cohomology groups of K with coefficients G are defined by

$H^n (K: G) = H^n(Hom (C_* (K), G)$

Lemma:

If $(S_*(X), \partial)$ is the singular complex of a space X then for every abelian group G

$$\mathbf{0} \longrightarrow \mathbf{Hom}\,(S_0(X), G) \xrightarrow{\partial_1^{\#}} \mathbf{Hom}\,(S_1(X), G) \xrightarrow{\partial_2^{\#}} \mathbf{Hom}\,(S_2(X), G) \longrightarrow \ldots$$

is a complex.

Proof:

Given that $(S_*(X), \partial)$ is the singular complex of a space X.

To prove that $(S_*(X), \partial)$ is complex.

It is enough to prove that $\partial^{\#}_{n+1}\ \ \partial^{\#}_{n} = 0$

Since $S_{n+1}(X)$ is generated by all (n+1) simplex σ it sufficient to show that $\partial\partial\sigma = 0$ by using the definition of boundary.

Now,

$$\partial\partial\sigma = \partial \textstyle\sum_j (-1)^{j}\, \sigma\, \varepsilon_j^{n+1}$$

$$= \textstyle\sum_{j,k} (-1)^{j+k}\, \sigma\, \varepsilon_j^{n+1}\varepsilon_k^{n}$$

$$= \textstyle\sum_{j<k} (-1)^{j+k}\, \sigma\, \varepsilon_j^{n+1}\varepsilon_k^{n} + \textstyle\sum_{k<j} (-1)^{j+k}\, \sigma\, \varepsilon_j^{n+1}\varepsilon_k^{n}$$

In the second sum

Let P = K, q = j-1

$$\partial\partial\sigma = \textstyle\sum_{j\le k} (-1)^{j+k}\, \sigma\, \varepsilon_j^{n+1}\varepsilon_k^{n} + \textstyle\sum_{p\le q} (-1)^{p+q+1}\, \sigma\, \varepsilon_p^{n+1}\varepsilon_q^{n}$$

$$= (-1)^{0+1}\, \sigma\, \varepsilon_0^{n+1}\varepsilon_1^{n} + (-1)^{0+1+1}\, \sigma\, \varepsilon_0^{n+1}\varepsilon_1^{n}$$

$$= -\sigma\, \varepsilon_0^{n+1}\varepsilon_1^{n} + \sigma\, \varepsilon_0^{n+1}\varepsilon_1^{n}$$

$\therefore \partial\partial\sigma = 0$

In general, $\partial^{\#}_{n+1}\, \partial^{\#}_{n} = 0$

Thus $(S_*(X), \partial)$ is complex.

Lemma:

If f: X → Y is continuous then for every n≥0

i) $f_{\#}\, (Z_n(X)) \subset Z_n(Y)$
ii) $f_{\#}\, (B_n(X)) \subset B_n(Y)$

Proof:

Given that the map f: X → Y is continuous

To prove that: i) $f_{\#}(Z_n(X)) \subset Z_n(Y)$

ii) $f_{\#}(B_n(X)) \subset B_n(Y)$

i) $f_{\#}(Z_n(X)) \subset Z_n(Y)$

If $\propto \in Z_n(X)$ then $\partial \propto = 0$

$\therefore \partial f_{\#} \propto = f_{\#} \propto$

$= f_{\#}(0) \quad = 0$

Which gives $f_{\#} \propto \in \ker \partial_n = Z_n(Y)$

$f_{\#} \propto \in (Y)$

$f_{\#}(Z_n(X)) \in Z_n(Y)$

$\therefore f_{\#}(Z_n(X)) \subset Z_n(Y)$

ii) $f_{\#}(B_n(X)) \subset B_n(Y)$

if $\beta \in B_n(X)$ then $\beta = \partial U$

For some $U \in S_{n+1}(X)$ and

$f_{\#} \beta = f_{\#} \partial U$

$= \partial f_{\#} U \in$ in ∂_{n+1}

$= B_n(Y)$

$f_{\#} \beta \in B_n(Y)$

$f_{\#}(B_n(X)) \in B_n(Y)$

$\therefore f_{\#}(B_n(X)) \subset B_n(Y)$

Hence proved.

Theorem:

A complex (S_*, ∂) is an exact sequence is and only if $H_n(S_*, \partial) = 0$ for every n.

Proof:

Given that (S_*, ∂) is an exact sequence

To prove that $H_n(S_*, \partial) = 0$

The ^{n}th homology group of this complex is

$$H_n (S_*, \partial) = \frac{Zn\ (S_*,\partial)}{Bn\ (S_*,\partial)}$$

By definition of exact sequence means im = ker

That is $Z_n = B_n$

Hence $H_n (S_*, \partial) = 0$

Conversely,

Assume that $H_n (S_*, \partial) = 0$

To prove that A complex (S_*, ∂) is an exact sequence.

The n^{th} homology of this complex

$$H_n (S_*, \partial) = \frac{Zn\ (S_*,\partial)}{Bn\ (S_*,\partial)}$$

By hypothesis $H_n (S_*, \partial) = 0$

Thus $Z_n = B_n$ if and only if Ker ∂_n = im ∂_{n+1}

Hence a complex (S_*, ∂) is an exact sequence

CONCLUSION

This aim of this is to determine cohomology group in algebraic topology. de Rham cohomology, the complex exact sequence theorem for by using commutative diagram are present with example we hope this theory will help to the analysis and understanding of these topics.

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